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Thesis

CONTINUED FRACTIONS

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Purpose of the Thesis

The writings which include the arithmetic and analytic theories of continued fractions are uncombined in one volume; this thesis is an attempt to gather, correlate, arrange, and, whenever possible, to extend this material to further the study of continued fractions. Numerous texts, periodicals and other publications were consulted. Although the thesis does not include all the material on continued fractions, it does include the more important works; and it can be considered complete from the point of view of the beginner and advanced student.

History of Continued Fractions

The theory and study of continued fractions seems to have arisen in connection with two classical problems--that of finding the approximate values of radicals and the evaluation of π.

Although many of the ancient mathematicians had occasion to develop rational numbers into continued fractions, the arithmetic theory of the subject may be said to have begun with Rafael Bombelli (born c.1530) and the publishing of his book of algebra in Bologna in 1572, Bombelli, with the aid of continued fractions, devised a method of approximating the value of square roots; but it was left to Pietro Cataldi (1548-1626) to simplify the process and it was Cataldi who first employed the more modern notation for continued fractions.

It was not until 1625, when Daniel Schwenter (1585-1636) investigated the theory of continued fractions, that any material contribution towards determining the approximants of continued fractions was made. He devoted his attention to the reduction of fractions involving large numbers and helped to determine the rules now in use for calculating the successive approximants. Christiaan Huygens (1629-1695) and John Wallis (1616-1703) also explored this field, the latter dis-

covering the general rule which combines the terms of the approximants. It was Wallis who termed this new mathematical theory continued fractions.

The relationship which first joined continued fractions and π was derived in 1658 by Lord Brouncker (c.1620-1684) when he transformed Wallis' product

$$\frac{\pi}{4} = \frac{2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdots}{3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdots}$$

into the continued fraction

$$\frac{\pi}{4} = \cfrac{1}{1+\cfrac{1}{2+\cfrac{3}{2+\cfrac{5}{2+\cdots}}}} \quad .$$

The theory of continued fractions received its greatest development in the eighteenth century with Leonhard Euler (1707-1783). Euler showed that any quadratic surd can be represented by a simple periodic continued fraction. He also evolved $\underline{e}$ as a continued fraction of the form

$$e = 2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cdots}}}}} \quad .$$

The converse of Euler's theorem, that any simple periodic continued fraction represents a solution of a quadratic equation, was proved to be true by Joseph

Lagrange (1736-1813). Continued fractions were applied by Lagrange in 1767 to the numerical solution of algebraic equations when he invented a method of successive approximations.

The expansion of π into the simple continued fraction

$$\pi = 3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\ddots}}}}}}}}$$

was discovered by Johann Lambert (1728-1777) in 1761 and is of great historical importance, The non-periodicity of this fraction shows conclusively that the number π is not a root of a quadratic equation; and this is just another way of saying that the quadrature of the circle cannot be accomplished by means of straight-edge and compass only.

The analytic theory of continued fractions, that is, continued fractions applied to analysis: theory of equations, power series, orthogonal polynomials, etc., did not come into its own until T.J. Stieljes published his celebrated memoir, <u>Recherches sur les frations continues</u>, in 1894. In this exposition, Stieltjes developed a general theory of continued fractions covering questions of convergence and connection with definite

integrals and divergent power series. In order to complete the theory, he had to extend the customary notion of the integral and so developed the species of integral since named after him. It should be mentioned, however, that Edmond Laguerre (1834-1886) had, in 1879, converted a divergent power series into a convergent continued fraction.

Continued fractions proved to be of great interest to many of the mathematicians of the nineteenth and early twentieth centuries. Included among these are Karl Gauss (1777-1855) who, in 1812, developed the quotient of two power series in the form of a continued fraction; P. Tschebycheff who generated the theory of orthogonal polynomials when he observed that partial denominators of a certain type of continued fractions form a sequence of orthogonal polynomials; A. Pringsheim and E. V. Van Vleck considered the question of convergence of continued fractions with complex elements; and more recently, H. S. Wall who has developed and compiled many interesting and new features concerning continued fractions.

As we shall soon see, the theoretical development of continued fractions is straight forward and interesting. The development of quadratic surds, π , *e*, and certain analytic functions into remarkably elegant continued fractions will, undoubtedly cause one to reflect and wonder upon the real essence and purpose of

continued fractions. Yes, one might ask, "What of continued fractions?" "Will they assume a more honored place in the realm of mathematical theories?" And in answer we may give Tobias Dantzig's inspiring observation:[1]

> "The conic section, invented in an attempt to solve the problem of doubling the altar of an oracle, ended by becoming the orbits followed by the planets in their courses about the sun. The imaginary magnitudes invented by Cardan and Bombelli describe in some strange way the characteristic features of alternating currents. The absolute differential calculus which originated as a fantasy of Riemann, became the mathematical vehicle for the theory of relativity. And the matrices which were a complete abstraction in the days of Cayley and Sylvester appear admirably adapted to the exotic situation exhibited by the quantum theory of the atom."

1. Scientific American, November 1948, Page 59.

Introduction to Continued Fractions

By way of introducing continued fractions, we will first develop rational approximations for the square root of 2.[1] This number being contained between 1 and 2, we set $\sqrt{2}=1+1/y$ where $y>1$. From this we draw $y=1/(\sqrt{2}-1)=\sqrt{2}+1$. But, since $\sqrt{2}=1+1/y$ we have $y=2+1/y$. Continuing in this manner, we obtain the infinite continued fraction

$$\sqrt{2} = 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}} \quad .$$

This is a special type of continued fraction; it is called simple because all the partial numerators are one (1), and periodic because the partial denominators repeat.

If we limit ourselves to one element, two elements, three, etc., of a continued fraction, we obtain a set of rational approximations, which are called approximants (or convergents). In the case of $\sqrt{2}$, the approximants are

$$1,\ 1\ 1/2,\ 1\ 2/5,\ 1\ 5/12,\ 1\ 12/29,\ 1\ 29/70,\ 1\ 70/169, \ldots \quad .$$

1. Tobias Dantzig, Number -- The Language of Science, Pages 155-156.

Two features make continued fractions particularly valuable. In the first place, a simple continued fraction always converges, and, in the second place, it has an oscillating character. In fact, we can break up the approximants into two groups by taking the 0th, 2nd, 4th, etc.; then the 1st, 3rd, 5th, etc. In the case of $\sqrt{2}$, we obtain the two asymptotic sequences:

1		1 2/5		1 12/29		1 70/169 ...	
	1 1/2		1 5/12		1 20/70 ...		.

The first (the even approximants) is continually increasing and has $\sqrt{2}$ for its upper bound, the other (the odd approximants) is continually decreasing and has $\sqrt{2}$ for its lower bound. This oscillating feature makes continued fractions invaluable for accurate approximations, for the error committed in stopping at any approximant can readily be estimated. The above material is rigorously developed in Chapter II.

Now let us formulate Felix Klein's geometric interpretation of continued fractions.[1] Confining our attention to positive numbers, let us mark all those in the positive quadrant of the XY plane (see figure 1) which have integral coordinates. A straight line from the origin to the point (a,b) has for its equation

$$x/y = a/b$$

and there are upon every such line $x/y=z$, where $z=a/b$ is rational, infinitely many integral points (ma, mb), where m is an arbitrary whole number. With the ex-

1. Felix Klein, Elementary Mathematics, pages 43-44.

ception of 0 itself, there is not a single integral point lying upon the irrational line $x/y=w$, where w is irrational (figure 1). Such a line makes a cut in the field of integral points by separating the points into two point sets, one lying to the right of the line and one to the left. If we inquire how these point sets converge toward our line $x/y=w$, we shall find a very simple relation to the continued fraction for w. By marking each point ($x=A_q$, $y=B_q$), corresponding to the approximant A_q/B_q, we see that the lines to these points approximate to the line $x/y=w$ better and better, alternately from the left and from the right, just as the numbers A_q/B_q, approximate the number w. Now, if we imagine pegs affixed at all the integral points, and wrap a tightly drawn string about the sets of pegs to the right and to the left of the w-line, then the vertices of the two convex string-polygons which bound our two point sets will be precisely the points (A_q, B_q) whose coordinates are the numerators and denominators of the successive approximants to w, the left polygon having the even approximants, the right one the odd. The representation in Figure 1 corresponds to the example

$$w = (\sqrt{5}-1)/2 = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}} \quad .$$

In this example, the first few vertices of the two polygons are:

left: $A_0=0$, $B_0=1$; $A_2=1$, $B_2=2$; $A_4=3$, $B_4=5$;...

right: $A_1=1$, $B_1=1$; $A_3=2$, $B_3=3$; $A_5=5$, $B_5=8$;... .

This explicit portrayal gives a new and graphic definition of a continued fraction.

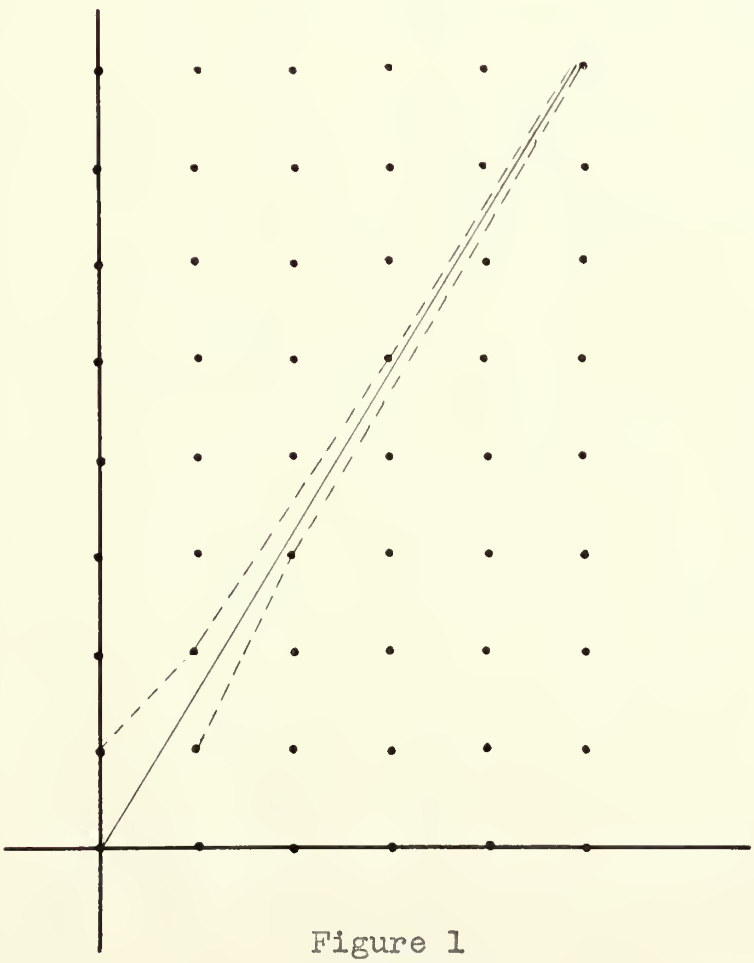

Figure 1

We have adopted the natural and explicit notation for a continued fraction. Other notations in more or less common use are:

$$b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots ,$$

$$b_0 + \frac{a_1 |}{| b_1} + \frac{a_2 |}{| b_2} + \frac{a_3 |}{| b_3} + \cdots ,$$

$$b_0 + \frac{a_1}{b_1} \dot{+} \frac{a_2}{b_2} \dot{+} \frac{a_3}{b_3} \dot{+} \cdots ,$$

$$b_0 + \underset{q=1}{\overset{q=\infty}{K}} \frac{a_q}{b_q} ,$$

$$b_0 + \left[\frac{a}{b} \right]_1^\infty .$$

CHAPTER I

General Theory and Definitions

1.1 By a continued fraction is meant an expression of the form

$$(1.1.1)\qquad b_0+\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\cdots\cfrac{a_q}{b_q+\cdots\cfrac{a_n}{b_n}}}}}\ ;$$

the primary interpretation of which is that a_1 is the numerator of a quotient whose denominator is all that lies under the line beneath a_1 and so on. If n is finite, we have a terminating or finite continued fraction; if n is infinite, the fraction is non-terminating or infinite.

1.2 In the most general case, the fractions

$$\frac{a_1}{b_1},\ \frac{a_2}{b_2},\ \ldots,\ \frac{a_q}{b_q},\ \ldots$$

may have numerators and denominators which can be any complex numbers. a_q/b_q is called the qth partial quotient.

1.3 The elements,

$$a_1,\ a_2,\ \ldots,\ a_q,\ \ldots$$
$$b_1,\ b_2,\ \ldots,\ b_q,\ \ldots\quad ,$$

are termed respectively the partial numerators and par-

tial denominators. a_q being the qth partial numerator and b_q the qth partial denominator. The elements may succeed each other without recurrence according to any law whatever. If they do recur, we have a recurring or periodic continued fraction.

1.4 The quantity,

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots \cfrac{a_q}{b_q}}},$$

is called the qth approximant. The numerator of the qth approximant is the qth numerator and it is denoted by A_q ; the qth denominator by B_q . We therefore have

$$\frac{A_0}{B_0} = \frac{b_0}{1},$$

$$\frac{A_1}{B_1} = b_0 + \frac{a_1}{b_1} = \frac{b_0 b_1 + a_1}{b_1},$$

$$\frac{A_2}{B_2} = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2}} = \frac{b_0 b_1 b_2 + b_0 a_2 + b_2 a_1}{b_1 b_2 + a_2},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot,$$

$$(1.4.1) \quad \frac{A_q}{B_q} = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots \cfrac{a}{b_q}}}$$

with

$$A_0 = b_0, \quad B_0 = 1\ ;$$

$$A_1 = b_0 b_1 + a_1, \quad B_1 = b_1\ ;$$

$$A_2 = b_0 b_1 b_2 + b_0 a_2 + b_2 a_1, \quad B_2 = b_1 b_2 + a_2\ ;$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

From these equalities it is seen that

$$A_2 = b_2 A_1 + a_2 A_0$$

$$B_2 = b_2 B_1 + a_2 B_0 \quad .$$

This suggests the following general recurrence formulas,

$$(1.4.2) \quad \begin{aligned} A_{q+1} &= b_{q+1} A_q + a_{q+1} A_{q-1} \\ B_{q+1} &= b_{q+1} B_q + a_{q+1} B_{q-1} \end{aligned} \quad q = 0, 1, 2, \ldots$$

with the initial conditions

$$A_{-1} = 1, \ B_{-1} = 0, \ A_0 = b_0, \ B_0 = 1,$$

which enable us to calculate the numerator and denominator of any approximant when the numerators and denominators of the two preceding approximants are known.

We then have the expression for the qth approximant

$$(1.4.3) \qquad \frac{A_q}{B_q} = \frac{b_q A_{q-1} + a_q A_{q-2}}{b_q B_{q-1} + a_q B_{q-2}} \quad .$$

1.5 Proof of (1.4.2) and (1.4.3)[1].

Let us assume that the formulas (1.4.2), (1.4.3) are true for the qth approximant. We observe from the definition of the qth approximant (1.4.1) that A_{q+1}/B_{q+1} is derived from A_q/B_q if we replace b_q by

1. Chrystal, Algebra, Volume II, page 405.

$b_q + a_{q+1}/b_{q+1}$. Hence, since A_{q-1}, B_{q-1}, A_{q-2}, B_{q-2} do not contain b_q, and since, by hypothesis,

$$\frac{A_q}{B_q} = \frac{b_q A_{q-1} + a_q A_{q-2}}{b_q B_{q-1} + a_q B_{q-2}}$$

it follows that

$$\frac{A_{q+1}}{B_{q+1}} = \frac{(b_q + a_{q+1}/b_{q+1})A_{q-1} + a_q A_{q-2}}{(b_q + a_{q+1}/b_{q+1})B_{q-1} + a_q B_{q-2}},$$

$$= \frac{(b_q b_{q+1} A_{q-1} + a_{q+1} A_{q-1} + a_q b_{q+1} A_{q-2})}{(b_q b_{q+1} B_{q-1} + a_{q+1} B_{q-1} + a_q b_{q+1} B_{q-2})},$$

$$= \frac{b_{q+1}(b_q A_{q-1} + a_q A_{q-2}) + a_{q+1} A_{q-1}}{b_{q+1}(b_q B_{q-1} + a_q B_{q-2}) + a_{q+1} B_{q-1}},$$

and by (1.4.2)

$$\frac{A_{q+1}}{B_{q+1}} = \frac{b_{q+1} A_q + a_{q+1} A_{q-1}}{b_{q+1} B_q + a_{q+1} B_{q-1}}$$

and the formulas are proved by induction.

1.6 The determinant of the equations (1.4.2) is

(1.6.1) $$\begin{vmatrix} A_{q-1} & A_q \\ B_{q-1} & B_q \end{vmatrix}$$

and we have by (1.4.2) that (1.6.1) is equal to

$$\begin{vmatrix} A_{q-1} & b_q A_{q-1} + a_q A_{q-2} \\ B_{q-1} & b_q B_{q-1} + a_q B_{q-2} \end{vmatrix}$$

$$= \begin{vmatrix} A_{q-1} & b_q A_{q-1} + a_q A_{q-2} - b_q A_{q-1} \\ B_{q-1} & b_q B_{q-1} + a_q B_{q-2} - b_q B_{q-1} \end{vmatrix}$$

$$= -a_q \begin{vmatrix} A_{q-2} & A_{q-1} \\ B_{q-2} & B_{q-1} \end{vmatrix}$$

and by repeated application of (1.4.2) we obtain the determinant formula[1]

(1.6.2) $$A_{q-1}B_q - A_qB_{q-1} = (-1)^q a_0a_1\cdots a_q$$

$$q = 0, 1, 2, \ldots$$

where a_0 must be taken equal to unity.

1.7 The Continued Fraction as a Product of Linear Fractional Transformations[2].

Let

$$\tau = \tau_q(w) = \frac{\alpha_q w + \beta_q}{\gamma_q w + \delta_q} \qquad \gamma_q \neq 0, \quad q = 0, 1, 2, \ldots$$

be an infinite sequence of linear fractional transformations of the variable w into the variable τ, and consider the product $\tau_0\cdots\tau_n(w)$ of the first $n+1$ of these transformations given by

$$\tau_0\tau_1(w) = \tau_0[\tau_1(w)],$$

$$\tau_0\tau_1\tau_2(w) = \tau_0\tau_1[\tau_2(w)], \ \ldots \quad .$$

If we write

$$\tau_q(w) = \frac{\gamma_q(\alpha_q w + \beta_q) + \alpha_q\delta_q - \alpha_q\delta_q}{\gamma_q(\gamma_q w + \delta_q)}$$

$$= \frac{\alpha_q(\delta_q + \gamma_q w) - (\alpha_q\delta_q - \beta_q\gamma_q)}{\gamma_q(\gamma_q w + \delta_q)}$$

$$= \frac{\alpha_q}{\gamma_q} - \frac{(\alpha_q\delta_q - \beta_q\gamma_q)}{\gamma_q(\gamma_q w + \delta_q)}$$

$$= \frac{\alpha_q}{\gamma_q} - \frac{\dfrac{(\alpha_q\delta_q - \beta_q\gamma_q)}{\gamma_q^2}}{\dfrac{\delta_q + \gamma_q w}{\gamma_q}}$$

$$= \frac{\alpha_q}{\gamma_q} - \frac{\dfrac{\Delta_q}{\gamma_q^2}}{\dfrac{\delta_q}{\gamma_q} + w} \qquad \Delta_q = \alpha_q\delta_q - \beta_q\gamma_q \ .$$

1. Wall, <u>Continued Fractions</u>, page 15.
2. Wall, <u>Continued Fractions</u>, pages 13-14.

We now have

$$\tau_0(w) = \frac{\alpha_0}{\gamma_0} - \frac{\frac{\Delta_0}{\gamma_0^2}}{\frac{\delta_0}{\gamma_0} + w}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\tau_n(w) = \frac{\alpha_n}{\gamma_n} - \frac{\frac{\Delta_n}{\gamma_n^2}}{\frac{\delta_n}{\gamma_n} + w}$$

and the required product

$$\tau_0\tau_1\cdots\tau_n(w) =$$

$$\frac{\alpha_0}{\gamma_0} - \cfrac{\frac{\Delta_0}{\gamma_0^2}}{\frac{\delta_0}{\gamma_0} + \frac{\alpha_1}{\gamma_1} - \cfrac{\frac{\Delta_1}{\gamma_1^2}}{\frac{\delta_1}{\gamma_1} + \frac{\alpha_2}{\gamma_2} - \cdots \cfrac{\frac{\Delta_{n-1}}{\gamma_{n-1}^2}}{\frac{\delta_{n-1}}{\gamma_{n-1}} + \frac{\alpha_n}{\gamma_n} - \cfrac{\frac{\Delta_n}{\gamma_n^2}}{\frac{\delta_n}{\gamma_n} + w}}}} .$$

If we put $w=\infty$ and then let $\underline{n}$ tend to ∞, the resulting infinite expression which is generated is an infinite continued fraction. In case at most a finite number of the quantities $\tau_0\tau_1\cdots\tau_n(\infty)$ are meaningless, and

$$\lim_{n\to\infty} \tau_0\tau_1\cdots\tau_n(\infty) = v$$

exists and is finite, then the continued fraction is said to converge, and $\underline{v}$ is called its value.

Using the simpler transformations

$$t_0(w) = b_0 + w \qquad q = 1, 2, 3, \ldots$$
$$t_q(w) = \frac{a_q}{b_q + w}$$

The continued fraction which is generated is

(1.7.1)
$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots}} \quad .$$

We observe that

$$t_0 t_1 \cdots t_n(0) = t_0 t_1 \cdots t_{n+1}(\infty)$$

for we have

$$\ddots + \frac{a_n}{b_n + 0} = \ddots + \cfrac{a_n}{b_n + \cfrac{a_{n+1}}{b_{n+1} + \infty}}$$

and the value of (1.1.1) is

$$\lim_{n\to\infty} t_0 t_1 \cdots t_n(0) = \lim_{n\to\infty} t_0 t_1 \cdots t_n(\infty) \quad .$$

The qth approximant is then given by

(1.7.2)
$$t_0 t_1 \cdots t_q(0) = \frac{A_q}{B_q} \quad .$$

1.8 Definition of Convergence.[1]

The continued fraction (1.1.1) is said to converge or to be convergent if at most a finite number of its denominators B_q vanish, and if the limit of its sequence of approximants,

(1.8.1)
$$\lim_{n\to\infty} \frac{A_n}{B_n} \quad ,$$

1. Wall, Continued Fractions, page 16.

exists and is finite. Otherwise, the continued fraction is said to diverge or to be divergent. The value of an infinite continued fraction is defined to be the limit (1.8.1) of its sequence of approximants. No value is assigned to a divergent continued fraction.

If the partial numerators a_q are all different from zero so that by (1.6.2) A_n and B_n cannot both vanish, then the existence of the finite limit (1.8.1) insures that but a finite number of the denominators B_n can vanish. Hence, in this important case, the continued fraction converges if (and only if) the limit (1.8.1) exists and is finite.

1.9 Definition of Uniform Convergence.

If the elements a_q and b_q of a continued fraction are functions of one or more variables over a certain domain D, then the continued fraction is said to converge uniformly over D if it converges for all values of the variable or variables in D, and if its sequence of approximants converges uniformly over D.

1.10 Equivalence Transformations[1].

It is often convernient to throw the continued fraction (1.1.1) into another form by means of a so-called equivalence transformation. This consists in multiplying numerators and denominators of suc-

1. Wall, Continued Fractions, pages 19-20

cessive fractions by numbers differenct from zero:

$$(1.10.1) \qquad b_0+\cfrac{c_1a_1}{c_1b_1+\cfrac{c_1c_2a_2}{c_2b_2+\cfrac{c_2c_3a_3}{c_3b_3+\cdots}}} \qquad (c_q \neq 0) \quad .$$

One may easily show by mathematical induction that this continued fraction has precisely the approximants of (1.1.1). In fact, the qth numerator and denominator of (1.10.1) are $c_1c_2\cdots c_qA_q$ and $c_1c_2\cdots c_qB_q$; respectively, where A_q and B_q are the qth numerator and denominator of (1.1.1). This can be readily verified by means of the fundamental recurrence formulas (1.4.2).

If, conversely, two continued fractions with nonvanishing partial numerators have a common sequence of approximants, then either can be transformed into the other by means of an equivalence transformation. In fact, if A'_q and B'_q are the qth numerator and denominator of one continued fraction, and A_q and B_q are those of the other, then there must exist constants $C_q \neq 0$ such that

$$(1.10.2) \qquad A'_q = C_qA_q \, , \quad B'_q = C_qB_q \qquad q = 1,2,3,\ldots \quad .$$

Let

$$A_q = b_q A_{q-1} + a_q A_{q-2} \qquad q = 1,2,3,\dots\ ,$$
$$B_q = b_q B_{q-1} + a_q B_{q-2}$$
$$A_{-1} = 1, \quad B_{-1} = 0, \quad A_0 = b\ , \quad B_0 = 1\ .$$

Then, since $A_{q-1}B_{q-2} - A_{q-2}B_{q-1} \neq 0$, by virtue of (1.6.2) we conclude that the elements a_q and b_q are uniquely determined by the A_q and B_q. Similarly, the elements of the other continued fraction are uniquely determined by the A'_q and B'_q.

Let

$$A'_q = b'_q A'_{q-1} + a'_q A'_{q-2} \qquad q = 1,2,3,\dots\ ,$$
$$B'_q = b'_q B'_{q-1} + a'_q B'_{q-2}$$
$$A'_{-1} = 1, \quad B'_{-1} = 0, \quad A'_0 = b_0, \quad B'_0 = 1$$

so that by (1.10.2)

$$A_q = b'_q \frac{C_{q-1}}{C_q} A_{q-1} + a'_q \frac{C_{q-2}}{C_{q-1}} \cdot \frac{C_{q-1}}{C_q} A_{q-2}$$
$$B_q = b'_q \frac{C_{q-1}}{C_q} B_{q-1} + a'_q \frac{C_{q-2}}{C_{q-1}} \cdot \frac{C_{q-1}}{C_q} B_{q-2}\ .$$

Here we must take $C_{-1} = C_0 = 1$.

Putting $c_q = \dfrac{C_q}{C_{q-1}}$ we have

$$b'_q = c_q b_q, \quad a'_q = c_{q-1} c_q a_q, \quad q = 1,2,3,\dots\ .$$

Thus, the two continued fractions are the same up to an equivalence transformation. We note the following important special cases. If $b_q \neq 0$, $q = 1,2,3,\dots$, and we take $c_q = 1/b_q$, then (1.10.1) takes on the form

(1.10.3)
$$b_0 + \cfrac{d_1}{1 + \cfrac{d_2}{1 + \cfrac{d_3}{1 + \ddots}}} \quad .$$

Likewise, if $a_q \neq 0$, $q = 1,2,3,\ldots$, and we take $c_0 = 1$, and determine the other c_q recurrently by the equations $c_{q-1} c_q a_q = 1$, $q = 1,2,3,\ldots$, then (1.10.1) takes the form

(1.10.4)
$$b_0 + \cfrac{1}{e_1 + \cfrac{1}{e_2 + \cfrac{1}{e_3 + \ddots}}} \quad .$$

CHAPTER II

Simple Continued Fractions[1]

2.1 We will, at first consider the finite simple continued fraction of the form

(2.1.1)
$$b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots \cfrac{1}{b_n}}}}$$

with the conditions

(2.1.2) b_1, ..., b_n are positive integers and b_0 may be either a positive or negative integer.

Since n is finite, the nth approximant A_n/B_n is the continued fraction itself.

2.2 The notation,

$$[b_0,\ b_1,\ b_2,\ \ldots\ ,\ b_n]\ ,$$

will be used to represent a continued fraction of the form (2.1.1).

This notation gives us the following relationships for the approximants:

$$[b_0] = b_0,$$

$$[b_0, b_1] = \frac{b_0 b_1 + 1}{b_1}\ ,$$

$$[b_0, b_1, b_2] = \frac{b_0 b_1 b_2 + b_0 + b_2}{b_1 b_2 + 1}\ ,$$

1. Hardy and Wright, The Theory of Numbers, Chapter X.

and generally,

$$[b_0, b_1, \ldots, b_q] = \frac{A_q}{B_q} \qquad (0 \leq q \leq n),$$

where the qth numerator is a polynomial in $b_0, b_1, \ldots, b_q$ and the qth denominator is a polynomial in $b_1, \ldots, b_q$.

It is plain that

$$[b_0, b_1] = b_0 + 1/b_1,$$

$$[b_0, b_1, \ldots, b_{q-1}, b_q] = [b_0, b_1 \ldots, b_{q-1} + 1/b_q],$$

$$[b_0, b_1, \ldots, b_q] = b_0 + \frac{1}{[b_1, b_2, \ldots, b_q]}$$

$$= [b_0, [b_1, \ldots, b_q]]$$

for $1 \leq q \leq n$. More generally we have

$$[b_0, b_1, \ldots, b_q] = [b_0, b_1, \ldots, b_{m-1}, [b_m, b_{m+1}, \ldots, b_q]]$$

for $1 \leq m \leq q \leq n$.

2.3 Approximants of (2.1.1).

The recurrence formulas (1.4.2) for the approximants of a continued fraction of the form (2.1.1) become

$$\begin{aligned} &A_{q+1} = b_{q+1} A_q + A_{q-1}, \\ (2.3.1) \qquad &B_{q+1} = b_{q+1} B_q + B_{q-1}, \qquad\qquad q = 0, 1, 2, \ldots \\ &A_{-1} = 1,\ B_{-1} = 0,\ A_0 = b_0,\ B_0 = 1; \end{aligned}$$

and (1.4.3) becomes

$$(2.3.2) \qquad \frac{A_q}{B_q} = \frac{b_q A_{q-1} + A_{q-2}}{b_q B_{q-1} + B_{q-2}}.$$

The determinant formula (1.6.2) also undergoes a slight change for the simple continued fraction and reduces to

$$(2.3.3) \quad A_{q-1}B_q - A_qB_{q-1} = (-1)^q \quad \text{or} \quad A_qB_{q-1} - A_{q-1}B_q = (-1)^{q-1} \quad . \qquad q = 0,1,2,\ldots$$

From these relationships we obtain the following corollaries:

Corollary 2.3.1

The approximants of (2.1.1), as calculated by formulas (2.3.1), are fractions at their lowest terms. For, if A_q and B_q , for example had any common factor, that factor would by (2.3.3) divide $(-1)^q$ exactly. Hence A_q is prime to B_q; and A_q/B_q is at its lowest terms.

e.g. If $\frac{A_q}{B_q} = \frac{m\cdot a}{m\cdot b} = \frac{a}{b}$ it follows

and
$$m\cdot b\cdot A_{q-1} - m\cdot a\cdot B_{q-1} = (-1)^q$$
$$b\cdot A_{q-1} - a\cdot B_{q-1} = (-1)^q \quad .$$

Corollary 2.3.2

$$\frac{A_q}{B_q} - \frac{A_{q-1}}{B_{q-1}} = \frac{(-1)^{q-1}}{B_{q-1}B_q} \, .$$

This is a result of (2.3.3).

Corollary 2.3.3

$$\frac{A_q}{B_q} = b_0 + \frac{1}{B_0 B_1} - \frac{1}{B_1 B_2} + \cdots + \frac{(-1)^{q-1}}{B_{q-1} B_q} .$$

For

$$\frac{A_q}{B_q} = \frac{A_0}{B_0} + \left(\frac{A_1}{B_1} - \frac{A_0}{B_0}\right) + \left(\frac{A_2}{B_2} - \frac{A_1}{B_1}\right) + \cdots + \left(\frac{A_q}{B_q} - \frac{A_{q-1}}{B_{q-1}}\right)$$

$$= \frac{A_0}{B_0} + \left(\frac{A_1 B_0 - A_0 B_1}{B_0 B_1}\right) + \left(\frac{A_2 B_1 - A_1 B_2}{B_1 B_2}\right)$$

$$+ \cdots + \left(\frac{A_q B_{q-1} - A_{q-1} B_q}{B_{q-1} B_q}\right)$$

$$= b_0 + \frac{1}{B_0 B_1} - \frac{1}{B_1 B_2} + \cdots + \frac{(-1)^{q-1}}{B_{q-1} B_q} ,$$

by (2.3.1) and (2.3.3).

Corollary 2.3.4

$$A_q B_{q-2} - A_{q-2} B_q = (-1)^q b_q .$$

For

$$A_q B_{q-2} - A_{q-2} B_q = (b_q A_{q-1} + A_{q-2}) B_{q-2} - A_{q-2}(b_q B_{q-1} + B_{q-2})$$

$$= (A_{q-1} B_{q-2} - A_{q-2} B_{q-1}) b_q$$

$$= (-1)^q b_q ,$$

by (2.3.1) with $q+1=q$ and by (2.3.3).

Corollary 2.3.5

$$\frac{A_q}{B_q} - \frac{A_{q-2}}{B_{q-2}} = \frac{(-1)^q b_q}{B_q B_{q-2}} .$$

This is a result of corollary 2.3.4.

Corollary 2.3.6

The even approximants x_{2q} continually increase in value, while the odd approximants x_{2q+1} continually decrease.

Let

$$x_q = \frac{A_q}{B_q}, \quad x = x_n = \frac{A_n}{B_n} .$$

Since every B_q is positive and all $b_1, \ldots, b_q > 0$, by corollary 2.3.5, $x_q - x_{q-2}$ has the sign of $(-1)^q$.

Corollary 2.3.7

Every odd approximant is greater than any even approximant.

By corollary 2.3.2, $x_q - x_{q-1}$ has the sign of $(-1)^{q-1}$.

Corollary 2.3.8

The value of the continued fraction is greater than that of any of its even approximants and less than that of any of its odd approximants (except that it is equal to the last approximant, whether this be even or odd).

$x = x_n = \frac{A_n}{B_n}$ is the greatest of the even, or the least of the odd approximants and A_n/B_n is the continued fraction itself.

Corollary 2.3.9

$B_q \geq B_{q-1}$ for $q \geq 1$, with inequality when $q>1$.

In the first place, $B_0 = 1$, $B_1 = b_1 \geq 1$.

Hence, for $q=1$; $B_1 = b_1 \geq B_0 = 1$.

For $q>1$; $B_q = b_q B_{q-1} + B_{q-2} > B_{q-1}$.

<u>Corollary 2.3.10</u>

$B_q \geq q$, with inequality when $q > 3$.

For

$q=1$ we have

$$B_1 \geq 1,$$

$q=2$

$$B_2 = b_2 B_1 + B_0 = b_2 b_1 + 1 \geq 2,$$

$q=3$

$$B_3 = b_3 B_2 + B_1 = b_3 (b_1 b_2 + 1) + b_1 \geq 3,$$

$q=4$

$$B_4 = b_4 B_3 + B_2 = b_4 [b_3 (b_1 b_2 + 1) + b_1] + b_1 b_2 + 1$$
$$= b_1 b_2 b_3 b_4 + b_4 + b_1 b_4 + b_1 b_2 + 1 > 4,$$

and from this we see

$$B_5 = b_5 B_4 + B_3 > B_4 + 1 > 5$$

and

$$B_q \geq B_{q-1} + B_{q-2} > B_{q-1} + 1 \geq q$$

for $q > 3$.

It is easily seen that the sequence of approximants of the continued fraction, [1,1,1, ... , 1], has the smallest values for their denominators. We have

$B_0 = 1 > q$, $\qquad B_1 = 1 = q$,

$B_2 = 2 = q$, $\qquad B_3 = 3 = 3$,

$B_4 = 5 > q$, $\qquad B_5 = 8 > q$,

. . . . . .

$B_q \geq q$ $\quad$ and $\quad$ $B_q \geq B_{q-1}$.

Therefore, corollaries 2.3.9 and 2.3.10 are necessarily true for all other simple continued fractions of form (2.1.1).

2.4 The Representation of a Rational Number by a Finite Simple Continued Fraction.

Any finite simple continued fraction $[b_0, b_1, \ldots, b_n]$ represents a rational number $x = x_n = A_n/B_n$. We now prove that, conversely, every positive rational number is representable by a simple continued fraction, and that, apart from one ambiguity, the representation is unique.

<u>Lemma 2.4.1</u>

If <u>x</u> is representable by a simple continued fraction with an odd (even) number of approximants, it is also representable by one with an even (odd) number.

For, if $b \geq 2$,

$$[b_0, b_1, \ldots, b_n] = [b_0, b_1, \ldots, b_n - 1, 1]$$

$$b_0 + \cfrac{1}{b_1 + \cfrac{}{\ddots \; \cfrac{1}{b}}} = b_0 + \cfrac{1}{b_1 + \cfrac{}{\ddots \; \cfrac{1}{b_n - 1 + \cfrac{1}{1}}}}$$

while, if $b_n = 1$,

$$[b_0, b_1, \ldots, b_{n-1}, 1] = [b_0, b_1, \ldots, b_{n-2}, b_{n-1}+1]$$

$$b_0 + \cfrac{1}{b_1 + \cdots + \cfrac{1}{b_{n-1} + \cfrac{1}{1}}} = b_0 + \cfrac{1}{b_1 + \cdots + \cfrac{1}{b_{n-2} + \cfrac{1}{b_{n-1}+1}}} .$$

For example

$$[2,2,3] = [2,2,2,1]$$

$$2 + \cfrac{1}{2 + \cfrac{1}{3}} = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1}}} .$$

We now introduce the notion of complete quotients which involves the following relationships:

(2.4.1) $$x = b_0' = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cdots + \cfrac{1}{b_n}}} ,$$

$$b_1' = b_1 + \cfrac{1}{b_2 + \cdots + \cfrac{1}{b_n}} ,$$

(2.4.2) $$b_g' = b_g + \cfrac{1}{b_{g+1} + \cdots + \cfrac{1}{b_n}} = [b_g, b_{g+1}, \ldots, b_n] ,$$

(2.4.3) $$b_g' = b_g + \frac{1}{b_{g+1}'} ,$$

(2.4.4) $$b_n' = b_n ,$$

where b' is called the (q+1)th complete quotient of the continued fraction $[b_0, \ldots, b_n]$.

Using the above notation, we have the following relation

(2.4.5) $$x = \frac{b_q' A_{q-1} + A_{q-2}}{b_q' B_{q-1} + B_{q-2}} \qquad (2 \leq q \leq n) \quad .$$

We will now prove (2.4.5) by induction:

For q=2

$$x = \frac{b_2' A_1 + A_0}{b_2' B_1 + B_0} \quad .$$

By (2.3.1)

$$A_0 = b_0, \quad A_1 = b_0 b_1 + 1,$$

$$B_0 = 1, \quad B_1 = b_1,$$

so

$$x = \frac{b_2'(b_0 b_1 + 1) + b_0}{b_2' b_1 + 1},$$

$$x = b_0 + \frac{b_2'}{b_2' b_1 + 1},$$

$$x = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2'}} \quad .$$

Assume true for q=q, then

$$x = \frac{b_q' A_{q-1} + A_{q-2}}{b_q' B_{q-1} + B_{q-2}}$$

and by (2.4.3)

$$x = \frac{(b_q + 1/b'_{q+1}) A_{q-1} + A_{q-2}}{(b_q + 1/b'_{q+1}) B_{q-1} + B_{q-2}},$$

$$x = \frac{(b_q b'_{q+1} + 1) A_{q-1} + b'_{q+1} A_{q-2}}{(b_q b'_{q+1} + 1) B_{q-1} + b'_{q+1} B_{q-2}},$$

$$x = \frac{b'_{q+1}(b_q A_{q-1} + A_{q-2}) + A_{q-1}}{b'_{q+1}(b_q B_{q-1} + B_{q-2}) + B_{q-1}} .$$

By (2.3.1)

$$x = \frac{b'_{q+1} A_q + A_{q-1}}{b'_{q+1} B_q + B_{q-1}}$$

and (2.4.5) is proved.

Lemma 2.4.2

$b_q = [b'_q]$, the integral part of b'_q, except that $b_{n-1} = [b'_{n-1}] - 1$ when $b_n = 1$. (Here the use of brackets around one element defines the integral part of the element).

If n = 0, then

$$b_0 = b'_0 = [b'_0].$$

If $n > 0$, then

$$b'_q = b_q + \frac{1}{b'_{q+1}} \qquad (0 \leq q \leq n-1) .$$

Now,

$$b'_{q+1} > 1 \qquad (0 \leq q \leq n-1)$$

except that

$$b'_{q+1} = 1$$

when $q=n-1$ and $b_n=1$.

Hence

(2.4.6) $$b_q < b'_q < b_q + 1 \qquad (0 \leq q \leq n-1)$$

and

$$b_q = [b'_q] \qquad (0 \leq q \leq n-1)$$

except in the case specified. And in any case

(2.4.7) $$b_n = b'_n = [b'_n].$$

Lemma 2.4.3

If two simple continued fractions $[a_0, a_1, \ldots, a_n]$, $[b_0, b_1, \ldots, b_m]$ have the same value $\underline{x}$, and $a_n > 1$, $b_m > 1$, then $m = n$ and the fractions are identical.

(When we say that two simple continued fractions are identical we mean that they are formed by the same sequence of partial denominators.)

By Lemma 2.4.2, $a_0 = [x] = b_0$. Let us suppose that the first $\underline{q}$ partial denominators in the continued fractions are identical, and that a'_q, b'_q are the $(q+1)$th complete quotients. Then

$$x = [a_0, a_1, \ldots, a_{q-1}, a'_q] = [a_0, a_1, \ldots, a_{q-1}, b'_q] .$$

If $q=1$, then

$$a_0 + \frac{1}{a'_1} = a_0 + \frac{1}{b'_1} ,$$

$a'_1 = b'_1$, and therefore, by Lemma 2.4.2, $a_1 = b_1$. If $q > 1$, then, by Lemma 2.4.1

$$x = \frac{a'_q A_{q-1} + A_{q-2}}{a'_q B_{q-1} + B_{q-2}} = \frac{b'_q A_{q-1} + B_{q-2}}{b'_q B_{q-1} + B_{q-2}} ,$$

and simplifying and factoring we have

$$(a'_q - b'_q)(A_{q-1} B_{q-2} - A_{q-2} B_{q-1}) = 0 .$$

But

$$A_{q-1} B_{q-2} - A_{q-2} B_{q-1} = (-1)^{q-2} , \text{ by } (2.3.3),$$

so $a'_q = b'_q$. It follows from Lemma 2.4.2 that $a_q = b_q$.

Let us suppose that $n \leq m$. Then our argument shows that $a_q = b_q$ for $q \leq n$. If $m > n$, then

$$x = \frac{A_n}{B_n} = [a_0, a_1, \ldots, a_n] = [a_0, a_1, \ldots, a_n, b_{n+1}, \ldots, b_m]$$

$$\frac{A_n}{B_n} = \frac{b'_{n+1} A_n + A_{n-1}}{b'_{n+1} B_n + B_{n-1}}$$

by (2.4.5); or after simplification

$$A_n B_{n-1} - A_{n-1} B_n = 0$$

which is false. Hence $m = n$ and the fractions are identical.

<u>Lemma 2.4.4</u> The Continued Fraction Algorithm.

Let *x* be an real number, and let $b_0 = [x]$. Then $x = b_0 + \beta_0$ with $0 \leq \beta_0 < 1$. If $\beta_0 \neq 0$, we can write

$$1/\beta_0 = b'_1, \quad [b'_1] = b_1, \quad b'_1 = b_1 + \beta_1, \quad (0 \leq \beta_1 < 1).$$

If $\beta_1 \neq 0$,

$$1/\beta_1 = b'_2 = b_2 + \beta_2, \qquad (0 \leq \beta_2 < 1),$$

and so on.

Since $b'_q = 1/\beta_{q-1} > 1$,

$$b_q \geq 1, \text{ for } q \geq 1.$$

Thus

$$x = [b_0, b'_1] = [b_0, b_1 + 1/b'_2] = [b_0, b_1, b'_2] = [b_0, b_1, b_2, b'_3]$$

and so on, where $b_0, b_1, \ldots$ are integers and $b_1 > 0, b_2 > 0, \ldots$.

The system of equations

$$
\begin{array}{ll}
x = b_0 + \beta_0 & (0 \leq \beta_0 < 1), \\
1/\beta_0 = b_1' = b_1 + \beta_1 & (0 \leq \beta_1 < 1), \\
1/\beta_1 = b_2' = b_2 + \beta_2 & (0 \leq \beta_2 < 1), \\
. \quad . \quad . \quad . \quad . \quad . \quad . \quad . & , \\
1/\beta_{q-1} = b_q' = b_q + \beta_q & (0 \leq \beta_q < 1)
\end{array}
$$

is known as the continued fraction algorithm. The algorithm continues so long as $\beta_q \neq 0$. If we eventually reach a value of q, say n, for which $\beta_n = 0$, the algorithm terminates and

$$x = [b_0, b_1, b_2, \ldots, b_n].$$

In this case x is represented by a finite simple continued fraction, and is rational. The numbers b_q' are the complete quotients of the continued fraction.

$$x = b_0 + \beta_1 = b_0 + 1/(b_1 + \beta_2),$$

$$x = b_0 + \cfrac{1}{b_1 + \cdots + \cfrac{1}{b_n}} \;\circ$$

Example:

$$
\begin{aligned}
x &= 840/611 = 1 + 229/611 \;, \\
1/\beta_1 &= 611/229 = 2 + 153/229 \;, \\
1/\beta_2 &= 229/153 = 1 + 76/153 \;, \\
1/\beta_3 &= 153/76 = 2 + 1/76 \;, \\
1/\beta_4 &= 76 = 76 + 0 \;, \\
\beta_5 &= 0 \;.
\end{aligned}
$$

We then have

$$x = 1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{76}}}} \quad .$$

Theorem 2.4.1

Any rational number can be represented by a finite simple continued fraction.

If $\underline{x}$ is an integer, then $\beta_0 = 0$ and $x = b_0$. If $\underline{x}$ is not integral, then $x = h/k$ where $\underline{h}$ and $\underline{k}$ are integers and $k > 1$. Since

$$h/k = b_0 + \beta_0 \ , \quad h = b_0 k + \beta_0 k \ ,$$

b_0 is the quotient, and $k_1 = \beta_0 k$ the remainder, when $\underline{h}$ is divided by $\underline{k}$.

If $\beta_0 \neq 0$, then $b_1' = 1/\beta_0 = k/k_1$ and $k/k_1 = b_1 + \beta_1$, $k = b_1 k_1 + \beta_1 k_1$. Thus b_1 is the quotient, and $k_2 = \beta_1 k_1$ the remainder when $\underline{k}$ is divided by k_1. We thus obtain a series of equations

$$h = b_0 k + k_1 , \quad k = b_1 k_1 + k_2 , \quad k_1 = b_2 k_2 + k_3 , \ \ldots$$

continuing so long as $\beta_q \neq 0$, or, what is the same thing, so long as $k_{q+1} \neq 0$ for $k_{q+1} = \beta_q k_q$.

The non-negative integers k, k_1, k_2, ... form a strictly decreasing sequence and $k_{n+1} = 0$ for some $\underline{n}$. It follows that $\beta_n = 0$ for some $\underline{n}$, and that the continued fraction algorithm terminates. This proves Theorem 2.4.1.

The system of equations

$$h = b_0 k + k_1 \qquad (0 < k_1 < k),$$

$$k = b_1 k_1 + k_2 \qquad (0 < k_2 < k_1),$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad ,$$

$$k_{n-2} = b_{n-1} k_{n-1} + k_{n-3} \qquad (0 < k_{n-2} < k_{n-1}),$$

$$k_{n-1} = b_n k_n$$

is known as Euclid's algorithm. This is the process adopted in elementary arithmetic to determine the greatest common divisor k_n of <u>h</u> and <u>k</u>. It should be noticed that, if $h < k$, the first quotient b_0 will be zero.

Since $\beta_n = 0$, $b_n' = b_n$; also by Lemma 2.4.4

$$0 < 1/b_n = 1/b_n' = \beta_{n-1} < 1,$$

and so $b_n \geq 2$ -- since all b_n are integers. Hence the algorithm determines a representation of the type which was shown to be unique by Lemma 2.4.3. We may always make the variation of Lemma 2.4.1.

Summing up our results we obtain

<u>Theorem 2.4.2</u>

A rational number can be expressed as a finite simple continued fraction in just two ways, one with an even and the other with an odd number of approximants. In one form the last partial denominator is <u>1</u>, in the other it is greater than <u>1</u>.

2.5 The Difference Between the Fraction and its Approximants.

By (2.4.5)

$$x = \frac{b'_{q+1} A_q + A_{q-1}}{b'_{q+1} B_q + B_{q-1}} \qquad \text{for} \quad 1 \leq q \leq n-1.$$

$$x - \frac{A_q}{B_q} = \frac{b'_{q+1} A_q + A_{q-1}}{b'_{q+1} B_q + B_{q-1}} - \frac{A_q}{B_q}$$

$$x - \frac{A_q}{B_q} = \frac{A_{q-1} B_q - A_q B_{q-1}}{B_q (b'_{q+1} B_q + B_{q-1})} = - \frac{(A_q B_{q-1} - A_{q-1} B_q)}{B_q (b'_{q+1} B_q + B_{q-1})}$$

$$= \frac{(-1)^q}{B_q (b'_{q+1} B_q + B_{q-1})} \qquad \text{by (2.3.3)} \quad .$$

Also

$$x - \frac{A_0}{B_0} = x - b_0 = \frac{1}{b'_1} \quad .$$

If we write

(2.5.1) $\quad B'_1 = b'_1, \quad B'_q = b'_q B_{q-1} + B_{q-2} \qquad (1 < q \leq n)$

so that

$B'_n = b'_n B_{n-1} + B_{n-2} = b_n B_{n-1} + B_{n-2} = B_n$ by (2.4.7) and (2.3.1)

we obtain

<u>Theorem 2.5.1</u>

If $1 \leq q \leq n-1$, then

$$x - \frac{A_q}{B_q} = \frac{(-1)^q}{B_q B'_{q+1}} \quad .$$

This formula offers another proof to Corollary 2.3.8 .

Next, by (2.4.6)

$$b_{q+1} < b'_{q+1} < b_{q+1} + 1 \qquad \text{for} \quad q \le n-2$$

except

$$b'_{n-1} = b_{n-1} + 1 \qquad\qquad \text{if} \quad b = 1.$$

If we ignore this exceptional case for the moment, we have

$$B_1 = b_1 < b_1' < b_1 + 1 \le B_2$$

for by (2.3.1)

$$B_1 = b_1 \;, \quad B_2 = b_1 b_2 + 1 \;,$$

and

$$B'_{q+1} = b'_{q+1} B_q + B_{q-1} > b_{q+1} B_q + B_{q-1} = B_{q+1}$$

(2.5.2)

$$B'_{q+1} > B_{q+1} \qquad\qquad q < n-1$$

by (2.4.6) with $q < n-1$.

Also, since

$$b'_{q+1} < b_{q+1} + 1$$

and by (2.3.1)

$$B'_{q+1} < (b_{q+1} + 1)B_q + B_{q-1} = b_{q+1} B_q + B_{q-1} + B_q = B_{q+1} + B_q$$

$$B'_{q+1} < B_{q+1} + B_q \le b_{q+2} B_{q+1} + B_q = B_{q+2}$$

(2.5.3) $\quad B'_{q+1} < B_{q+2} \qquad\qquad 1 \le q \le n-2 \quad .$

By Theorem 2.5.1

$$B_q x - A_q = \frac{(-1)^q}{B'_{q+1}} \qquad .$$

It follows from this and (2.5.2) and (2.5.3) that

(2.5.4) $$\frac{1}{B_{q+2}} < \left| A_q - B_q x \right| < \frac{1}{B_{q+1}} \qquad q \leq n-2$$

while

(2.5.5) $$\begin{aligned} \left| A_{n-1} - B_{n-1} x \right| &= \frac{1}{B_n} \quad , \\ A_n - B_n x &= 0 \end{aligned}$$

since $B_n' = B_n$.

In the exceptional case, (2.5.1) must be replaced by

$$B_{n-1}' = (b_{n-1} + 1)B_{n-2} + B_{n-3} = B_{n-1} + B_{n-2} = B_n$$

and (2.5.4) becomes

$$\frac{1}{B_{q+2}} = \left| A_q - B_q x \right| < \frac{1}{B_{q+1}} \quad .$$

In any case (2.5.5) shows that $|A_q - B_q x|$ decreases steadily as <u>q</u> increases; since B_q increases steadily

$$\left| x - \frac{A_q}{B_q} \right|$$

decreases steadily.

We may sum up the most important of our conclusions in

<u>Theorem 2.5.2</u>

If $n > 1$, $q > 0$, then the differences

$$x - \frac{A_q}{B_q}, \quad B_q x - A_q$$

decrease steadily in absolute value as <u>q</u> increases. Also

$$B_q x - A_q = \frac{(-1)^q \delta_q}{B_{q+1}},$$

where $0 < \delta_q < 1$, $(1 \leq q \leq n-2)$, $\delta_{n-1} = 1$

and

$$\left| x - \frac{A_q}{B_q} \right| \leq \frac{1}{B_q B_{q+1}} < \frac{1}{B_q^2}$$

for $q \leq n-1$, with inequality in both places except when $q = n-1$.

2.6 Infinite Simple Continued Fractions.

We have considered so far only finite continued fractions; and these, when they are simple, represent rational numbers. In the theory of numbers, the chief interest of continued fractions lies in their application to the representation of irrationals and for this, infinite continued fractions are needed.

Suppose that $b_0, b_1, b_2, \ldots,$ is a sequence of integers with conditions (2.1.2) so that

$$x_q = [b_0, b_1, \ldots, b_q]$$

is for every <u>q</u>, a simple continued fraction representing a rational number x_q. If, as we shall prove in a

moment, x_q tends to a limit x when q tends to ∞, then it will be natural to say that the infinite simple continued fraction

(2.6.1) $$[b_0, b_1, b_2, \ldots ,]$$

converges to the value x, and to write

(2.6.2) $$x = [b_0, b_1, b_2, \ldots ,] \; .$$

Theorem 2.6.1

If $b_0, b_1, b_2, \ldots$, is a sequence of integers satisfying (2.1.2), then

$$x_q = [b_0, b_1, \ldots , b_q]$$

tends to a limit x when q tends to ∞ . We may express this more shortly as

Theorem 2.6.2

All infinite simple continued fractions are convergent. (See definition of convergence.)

We write

$$x_q = \frac{A_q}{B_q} = [b_0, b_1, \ldots , b_q],$$

and call these fractions the approximants to (2.6.1). We have to show that the approximants tend to a limit.

If $n \geq q$, the approximant x_q is an approximant to $[b_0, b_1, \ldots , b_q]$. Hence, by Corollary 2.3.6, the even approximants form an increasing and the odd convergents a decreasing sequence.

Every even approximant is less than x_1, by Corollary (2.3.7) so that the increasing sequence of even convergents is bounded above; and every odd convergent is greater than x_0, so that the decreasing sequence of odd convergents is bounded below. Hence, the even convergents tend to a limit α_1, and the odd convergents to a limit α_2, and $\alpha_1 \leq \alpha_2$.

Finally, by Corollaries (2.3.2) and (2.3.10), we have

$$\left|\frac{A_{2q}}{B_{2q}} - \frac{A_{2q-1}}{B_{2q-1}}\right| = \frac{1}{B_{2q}\,B_{2q-1}} \leq \frac{1}{2q(2q-1)} \longrightarrow 0$$

as $\underline{q}$ tends to ∞. We now can say that $\alpha_1 = \alpha_2 = x$ and the fraction (2.6.1) converges to $\underline{x}$.

Corollary 2.6.1

An infinite simple continued fraction is less than any of its odd approximants and greater than any of its even convergents.

2.7 The Representation of an Irrational Number by an Infinite Continued Fraction.

We call

$$b_q' = [b_q, b_{q+1}, \ldots ,]$$

the qth complete quotient of the continued fraction

$$x = [b_0, b_1, \ldots ,] .$$

Clearly

$$b_q' = \lim_{n\to\infty} [b_q, b_{q+1}, \ldots, b_n]$$
$$= b_q + \lim_{n\to\infty} \frac{1}{[b_{q+1}, \ldots, b_n]} = b_q + \frac{1}{b_{q+1}'},$$

and in particular

$$x = b_0' = b_0 + \frac{1}{b_1'} .$$

Also

$$b_q' > b_q, \quad b_{q+1}' > b_{q+1} > 0, \quad 0 < \frac{1}{b_{q+1}'} < 1;$$

and

$$b_q = [b_q'] .$$

<u>Lemma 2.7.1</u>

If $[b_0, b_1, b_2, \ldots,] = x$, then

$$b_0 = [x], \quad b_q = [b_q'] \qquad (q \geq 0).$$

From this we deduce, as in Lemma 2.4.3,

<u>Lemma 2.7.2</u>

Two infinite simple continued fractions which have the same value are identical.

We now return to the continued fraction algorithm of Lemma 2.4.4. If $\underline{x}$ is irrational the process cannot terminate. Hence it defines an infinite sequence of integers $b_0, b_1, b_2, \ldots$, and as before

$$x = [b_0, b_1'] = [b_0, b_1, b_2'] = \ldots =$$
$$= [b_0, b_1, b_2, \ldots, b_q, b_{q+1}'] ,$$

where

$$b_{q+1}' = b_{q+1} + \frac{1}{b_{q+2}'} > b_{q+1} .$$

Hence

$$x = \frac{b'_{q+1} A_q + A_{q-1}}{b'_{q+1} B_q + B_{q-1}}$$

by (2.4.5) and so, as in section 2.5,

$$x - \frac{A_q}{B_q} = \frac{A_{q-1}B_q - A_q B_{q-1}}{B_q(b'_{q+1} B_q + B_{q-1})} = \frac{(-1)^q}{B_q(b'_{q+1} B_q + B_{q-1})},$$

$$\left| x - \frac{A_q}{B_q} \right| < \frac{1}{B_q(b_{q+1} B_q + B_{q-1})} = \frac{1}{B_q B_{q+1}} \leq \frac{1}{q(q+1)} \to 0$$

when q tends to ∞. Thus

$$x = \lim_{q\to\infty} \frac{A_q}{B_q} = [b_0, b_1, \ldots, b_q, \ldots,],$$

and the algorithm leads to the continued fraction whose value is x, and which is unique by Lemma 2.7.2.

Corollary 2.7.1

Every irrational number can be expressed in just one way as an infinite simple continued fraction.

Incidentally, we see that the value of an infinite simple continued fraction is necessarily irrational, since the algorithm would terminate if x were rational.

We define

$$B'_q = b'_q B_{q-1} + B_{q-2}$$

as in (2.5.1). Repeating the argument of that section, we obtain

Theorem 2.7.1

The results of Theorem 2.5.1 and Theorem 2.5.2 hold also (except for the references to q) for infinite continued fractions.

In particular

$$\left| x - \frac{A_q}{B_q} \right| < \frac{1}{B_q B_{q+1}} < \frac{1}{B_q^2} \quad .$$

We have thus shown that the main theorems associated with finite simple continued fractions are also applicable to infinite simple continued fractions. We can therefore conclude the following relationships for all simple continued fractions:

1. The even approximants form an increasing sequence of rational fractions continually approaching to the value of the whole continued fraction; and the odd approximants form a decreasing sequence having the same property.

2. The difference of the value of the continued fraction and the qth approximant is less than $1/B_q B_{q+1}$.

3. In order to obtain a good approximation to a continued fraction, it is advisable to take that approximant whose corresponding partial denominator immediately precedes a very much larger partial denominator. For, if the next denominator be large, there is a sudden

increase in B_{q+1} , so that $1/B_q B_{q+1}$ is a very small fraction which is to our advantage since

$$\left| x - \frac{A_q}{B_q} \right| < \frac{1}{B_q B_{q+1}} \quad .$$

Thus we have shown that the method of continued fractions possesses the two most important advantages that any system of numerical calculation can have; namely, it furnishes a regular sequence of rational approximations to the quantity to be evaluated; and the error committed by arresting the approximation at any step can at once be estimated.

Example:[1]

The continued fraction which represents the ratio of the circumference of a circle to the diameter is

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}} \quad .$$

By using the fundamental recurrence formulas for a simple continued fraction (2.1) we find that the successive approximants are

$$\frac{3}{1}\ ,\ \frac{22}{7}\ ,\ \frac{333}{106}\ ,\ \frac{355}{113}\ ,\ \frac{103993}{33102}\ ,\ \ldots, \quad .$$

1. Chrystal, Algebra, Volume II, page 415.

Two of these; namely, 22/7 and 355/113 are distinguished beyond the others by preceding large partial denominators, 15 and 292. The first of them, 22/7, was given by Archimedes (212 B.C.). The second, 355/113, was given by Adrian Metius (published by his son, 1640 A.D.). It is in great favor, not only on account of its accuracy, but because it can easily be remembered as consisting of the first three odd numbers each repeated twice in a certain succession. For 355/113, we have

$$\left| \pi - \frac{355}{113} \right| < \frac{1}{113 \cdot 33102} \quad ,$$

$$\left| \pi - \frac{355}{113} \right| < 0.0000002673 \quad .$$

That is, 355/113 is accurate to the 6th decimal place. In fact

$$\begin{array}{rl} \pi & = 3.14159265358\ldots \\ \underline{355/113} & \underline{= 3.14159292035\ldots} \\ \text{Difference} & = 0.00000026677\ldots \quad . \end{array}$$

2.8 Simple Periodic Continued Fractions.

A simple periodic continued fraction is an infinite continued fraction in which $b_l = b_{l+k}$ for a fixed positive $\underline{k}$ and all $l \geq L$. The set of partial denominators

$$b_L, b_{L+1}, \ldots, b_{L+k-1}$$

is called the period. This type of continued fraction may be written

$$[b_0, b_1, \ldots, b_{L-1}, \overset{*}{b}_L, b_{L+1}, \ldots, \overset{*}{b}_{L+k-1}]$$

where the ** indicate the beginning and end of the cycle of partial denominators.

For example, it will be shown that

$$\sqrt{13} = [3, \overset{*}{1}, 1, 1, 1, \overset{*}{6}]$$

$$= 3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cdots}}}}}}}}}} \quad .$$

Theorem 2.8.1

A periodic continued fraction is a quadratic surd, i.e. an irrational root of a quadratic equation with integral coefficients.

Let

$$x = [b_0, b_1, \dots, b_{L-1}, \overset{*}{b}_L, b_{L+1}, b_{L+2}, b_{L+3}, \dots, \overset{*}{b}_{L+k-1}]$$

and if b_L' is the Lth complete quotient, we have

$$b_L' = [b_L, b_{L+1}, \dots, b_{L+k-1}, b_L, b_{L+1}, \dots,]$$

$$= [b_L, b_{L+1}, \dots, b_{L+k-1}, b_L']$$

and by the fundamental recurrence formulas,

$$b_L' = \frac{A\, b_L' + A'}{B\, b_L' + B'}$$

and simplifying

$$(2.8.1) \qquad Bb_L'^2 + (B'-A)\, b_L' - A' = 0$$

where A'/B' is the penultimate approximant and A/B the last approximant to

$$[b_L, b_{L+1}, \dots, b_{L+k-1}] \; .$$

But

$$x = \frac{A_{L-1} b_L' + A_{L-2}}{B_{L-1} b_L' + B_{L-2}}$$

from which

$$b_L' = \frac{A_{L-2} - B_{L-2}x}{B_{L-1}x - A_{L-1}} \; .$$

If we substitute for b_i' in (2.8.1), and clear of fractions, we obtain an equation

(2.8.2) $$ax^2 + bx + c = 0$$

with integral coefficients. Since $\underline{x}$ is irrational by section 2.7, $b^2 - 4ac \neq 0$ and we have our result.

The converse of the theorem is also true and we have

<u>Theorem 2.8.2</u>

The continued fraction which represents a quadratic surd is periodic.

A quadratic surd satisfies a quadratic equation with integral coefficients, which we may write in the form (2.8.2). If

$$x = [b_0, b_1, \ldots, b_q, \ldots,],$$

then

$$x = \frac{A_{q-1} b_q' + A_{q-2}}{B_{q-1} b_q' + B_{q-2}},$$

and if we substitute this in (2.8.2) we obtain

(2.8.3) $$\alpha_q b_q'^2 + \beta_q b_q' + \gamma_q = 0,$$

where

$$\alpha_q = a\, A_{q-1}^2 + b\, A_{q-1} B_{q-1} + c\, B_{q-1}^2,$$
$$\beta_q = 2aA_{q-1} A_{q-2} + b(A_{q-1} B_{q-2} + A_{q-2} B_{q-1}) + 2cB_{q-1} B_{q-2},$$
$$\gamma_q = aA_{q-2}^2 + bA_{q-2} B_{q-2} + cB_{q-2}^2 .$$

If

$$\alpha_q = aA_{q-1}^2 + bA_{q-1}B_{q-1} + cB_{q-1}^2 = 0 \; ,$$
$$= a\frac{A_{q-1}^2}{B_{q-1}^2} + b\frac{A_{q-1}}{B_{q-1}} + c = 0 \; .$$

Then (2.8.2) has the rational root A_{q-1}/B_{q-1}, but this is impossible since x is irrational. Hence $\alpha_q \neq 0$ and

$$\alpha_q y^2 + \beta_q y + \gamma_q = 0$$

is an equation one of whose roots is b_q' . A little calculation shows that

$$(2.8.3) \qquad \beta_q^2 - 4\alpha_q\gamma_q = (b^2-4ac)(A_{q-1}B_{q-2}-A_{q-2}B_{q-1})^2$$
$$= (b^2-4ac) \qquad \text{by } (2.3.3) \; .$$

By theorem 2.7.1

$$A_{q-1} = xB_{q-1} + \frac{\delta_{q-1}}{B_{q-1}} \qquad (|\delta_{q-1}|<1) \; .$$

Hence

$$\alpha_q = a(xB_{q-1} + \frac{\delta_{q-1}}{B_{q-1}})^2 + bA_{q-1}(xB_{q-1} + \frac{\delta_{q-1}}{B_{q-1}}) + cB_{q-1}^2$$
$$= (ax^2 + bx + c)B_{q-1}^2 + 2ax\delta_{q-1} + a\frac{\delta_{q-1}^2}{B_{q-1}^2} + b\delta_{q-1}$$
$$= 2ax\delta_{q-1} + a\frac{\delta_{q-1}^2}{B_{q-1}^2} + b\delta_{q-1} \; ,$$

and

$$|\alpha_q| < 2|ax| + |a| + |b| \; .$$

Next, since $\gamma_q = \alpha_{q-1}$

$$|\gamma_q| < 2|ax| + |a| + |b| \; .$$

And, by (2.8.3)

$$\beta_n^2 \le 4\,|\alpha_q \gamma_q| + |b^2 - 4ac|$$
$$< 4\,(2\,|ax| + |a| + |b|\,)^2 + |b^2 - 4ac| \;.$$

Hence the absolute values of α_q, β_q, γ_q are less than numbers independent of $\underline{q}$.

It follows that there are only a finite number of different triplets $(\alpha_q, \beta_q, \gamma_q)$; and we can find a triplet (α, β, γ) which occurs at least three times, say as $(\alpha_{q_1}, \beta_{q_1}, \gamma_{q_1})$, $(\alpha_{q_2}, \beta_{q_2}, \gamma_{q_2})$ and $(\alpha_{q_3}, \beta_{q_3}, \gamma_{q_3})$. Hence b'_{q_1}, b'_{q_2}, b'_{q_3} are all roots of

$$\alpha y^2 + \beta y + \gamma = 0 \;,$$

and at least two of them must be equal. But if for example, $b'_{q_1} = b'_{q_2}$, then

$$b_{q_2} = b_{q_1} \,, \quad b \;\; = b_{q_1+1} \ldots \,,$$

and the continued fraction is periodic.

If the periodic continued fraction is of the form

$$x = [\overset{*}{b}_0, \ldots, \overset{*}{b}_k] = b_0 + \cfrac{1}{b_1 + \cdots + \cfrac{1}{b_k + \cfrac{1}{b_0 + \cdots}}}$$

is called a pure periodic continued fraction. A mixed periodic continued fraction is of the form

$$x = [b_0, b_1, \ldots, b_{L-1}, \overset{*}{b}_L, b_{L+1}, \ldots, \overset{*}{b}_{L+k-1}] \;.$$

If the periodic continued fraction is pure then

$$x = [b_0, \ \dots \ , b_k, b_0, \ \dots \ , b_k, \ \dots \ ,]$$

$$x = [b_0, \ \dots \ , b_k, \ x \ ]$$

$$x = \frac{A_k x + A_{k-1}}{B_k x + B_{k-1}} \tag{2.8.4}$$

Example:[1]

To convert $\sqrt{13}$ into a continued fraction.

We have, 3 being the greatest integer less than $\sqrt{13}$,

$$\sqrt{13} = 3 + (\sqrt{13}-3) = 3+\cfrac{1}{\cfrac{1}{\sqrt{13}-3}}$$

$$= 3+\cfrac{1}{\cfrac{\sqrt{13}+3}{4}} \qquad (1).$$

Again, since the greatest integer in $(\sqrt{13}+3)/4$ is 1, we have

$$\frac{\sqrt{13}+3}{4} = 1+\frac{\sqrt{13}-1}{4} = 1+\cfrac{1}{\cfrac{4}{\sqrt{13}-1}},$$

$$= 1+\cfrac{1}{\cfrac{\sqrt{13}+1}{3}} \qquad (2).$$

Similarly, we have

$$\frac{\sqrt{13}+1}{3} = 1+\frac{\sqrt{13}-2}{3} = 1+\cfrac{1}{\cfrac{3}{\sqrt{13}-2}},$$

$$= 1+\cfrac{1}{\cfrac{\sqrt{13}+2}{3}} \qquad (3);$$

1. Chrystal, <u>Algebra</u>, Volume II, Page 401.

$$\frac{\sqrt{13}+2}{3} = 1+\frac{\sqrt{13}-1}{3} = 1+\cfrac{1}{\cfrac{3}{\sqrt{13}-1}} ,$$

$$= 1+\cfrac{1}{\cfrac{\sqrt{13}+1}{4}} \qquad (4);$$

$$\frac{\sqrt{13}+1}{4} = 1+\frac{\sqrt{13}-3}{4} = 1+\cfrac{1}{\cfrac{4}{\sqrt{13}-3}} ,$$

$$= 1+\frac{1}{\sqrt{13}+3} \qquad (5);$$

$$\sqrt{13}+3 = 6+\sqrt{13}-3 = 6+\cfrac{1}{\cfrac{1}{\sqrt{13}-3}} ,$$

$$= 6+\cfrac{1}{\cfrac{\sqrt{13}+3}{4}} \qquad (6);$$

after which the process repeats itself.

From the equations (1)...(6) we obtain

$$\sqrt{13} = 3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\cdots}}}}}} = [3, \dot{1}, 1, 1, 1, \dot{6}] .$$

We see that the above process consists in finding the greatest integer in a series of irrational numbers, and in rationalizing the denominator of the reciprocal of the residue.

Example:[1]

To convert $\frac{\sqrt{3}-1}{2}$ into a continued fraction.

We have

$$\frac{\sqrt{3}-1}{2} = 0+\cfrac{1}{\cfrac{2}{\sqrt{3}-1}} \quad ,$$

$$= 0+\frac{1}{\sqrt{3}+1} \quad ;$$

$$\sqrt{3}+1 = 2+\sqrt{3}-1 = 2+\cfrac{1}{\cfrac{1}{\sqrt{3}-1}} \quad ,$$

$$= 2+\cfrac{1}{\cfrac{\sqrt{3}+1}{2}}$$

$$\frac{\sqrt{3}+1}{2} = 1+\frac{\sqrt{3}-1}{2} = 1+\cfrac{1}{\cfrac{2}{\sqrt{3}-1}} \quad ,$$

$$= 1+\frac{1}{\sqrt{3}+1} \quad ;$$

after which the quotients recur. We have, therefore,

$$\frac{\sqrt{3}+1}{2} = 0+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cdots}}}} = [0,\overset{*}{2},\overset{*}{1}] \quad .$$

Example:[2]

Evaluate $x = [\overset{*}{1},\ 2,\ \overset{*}{1}] = 1+\cfrac{1}{2+\cfrac{1}{1+\cdots}}$.

1. Chrystal, Algebra, Volume II, page 402.
2. Chrystal, Algebra, Volume II, page 431.

By (2.8.4)

$$x = \frac{A_k x + A_{k-1}}{B_k x + B_{k-1}} .$$

With k = 2, the two last convergents to $[\dot{1}, 2, \dot{1}]$ are $A_1/B_1 = 3/2$ and $A_2/B_2 = 4/3$.

Hence

$$x = \frac{4x + 3}{3x + 2} .$$

We therefore have

$$3x^2 - 2x - 3 = 0 ,$$

the positive root of which is

$$x = \frac{1 + \sqrt{10}}{3}$$

Example:[1]

Evaluate $y = [3, 4, \dot{1}, 2, \dot{1}]$.

The two last approximants to 3 + 1/4 are 3/1 and 13/4; and by example three above,

$$x = [\dot{1}, 2, \dot{1}] = \frac{1 + \sqrt{10}}{3} .$$

We have therefore

$$y = 3 + \cfrac{1}{4 + \cfrac{1}{x}}$$

$$y = \frac{A_n}{B_n} = \frac{b_n A_{n-1} + A_{n-2}}{b_n B_{n-1} + B_{n-2}} , \qquad (n=2)$$

$$= \frac{13x + 3}{4x + 1} ,$$

1. Chrystal, Algebra, Volume II, page 432.

$$= \frac{13[(1+\sqrt{10})/3] + 3}{4[(1+\sqrt{10})/3] + 1} \quad ,$$

$$= \frac{22 + 13\sqrt{10}}{7 + 4\sqrt{10}} \quad ,$$

$$= \frac{366 - 3\sqrt{10}}{111} \quad ,$$

$$y = \frac{122 - \sqrt{10}}{37} \quad .$$

CHAPTER III

General Continued Fractions and Convergence Theorems[1]

3.1 The general continued fraction is an infinite continued fraction of the form

(3.1.1)
$$x = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}}$$

where $a_1, a_2, a_3, \ldots, b_0, b_1, b_2, \ldots$, are real or complex elements. However, unless otherwise stated, we will deal with the situation where the elements are real and positive.

It should be pointed out that the continued fraction (3.1.1) may fail to have any definite meaning when an attempt is made to obtain its value by taking the limit of values of the successive approximants. For example

$$1 + \cfrac{1}{1 - \cfrac{1}{1 - \cfrac{1}{1 - \ddots}}}$$

has for its second approximant

$$1 + \frac{1}{1 - 1}$$

1. Chrystal, Algebra, Volume II, Chapter XXXIV.

and for its third approximant

$$1+\cfrac{1}{1-\cfrac{1}{1-1}} \quad .$$

We therefore cannot assign a value to a general continued fraction unless we have shown that the limit in question is finite and definite.

In cases where any difficulty regarding the meaning or convergency of the continued fraction arises, we regard (3.1.1) as representing the assemblage of approximants $\frac{A_0}{B_0}, \frac{A_1}{B_1}, \ldots, \frac{A_k}{B_k}$ whose denominators are constructed by means of the fundamental recurrence formulas (1.4.2).

3.2 We will now develop certain relationships between the approximants and elements of the general continued fraction. Some of these corollaries are analogous to those already established for simple continued fractions.[1]

A continued fraction of the form (3.1.1) may fall into one of the following two classes:

fractions of the first class,

$$b_0+\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+ \cdot_{\cdot_{\cdot}}}} \quad ;$$

1. Chrystal, <u>Algebra</u>, Volume II, Pages 464-5.

and fractions of the second class,

$$b_0 + \cfrac{a_1}{b_1 - \cfrac{a_2}{b_2 - \ddots}} \quad .$$

We will need to make use of the fundamental recurrence formulas

$$(3.2.1) \qquad \begin{aligned} A_q &= b_q A_{q-1} + a_q A_{q-2} \\ B_q &= b_q B_{q-1} + a_q B_{q-2} \end{aligned} \qquad q = 0,1,2, \ldots$$

with the initial conditions

$$A_{-1} = 1, \; B_{-1} = 0, \; A_0 = b_0, \; B_0 = 1 \quad ;$$

and the determinant formula

$$(3.2.2) \qquad A_{q-1} B_q - A_q B_{q-1} = (-1)^q a_0 a_1 a_2 \ldots a_q \qquad q = 0,1,2, \ldots$$

where $a_0 = 1$.

<u>Corollary 3.2.1</u>

In a continued fraction of the first class A_q and B_q are both positive; and, provided $b_q \geq 1$, each of them continually increases with <u>q</u>.

In a continued fraction of the second class, subject to the restriction $b_q - 1 \geq a_q$, A_q and B_q are positive, and each of them increases with <u>q</u>. (We must remember to substitute $-a_2, \ldots, -a_q$ for $a_2, \ldots, a_q$ when using (3.2.1) and other formulas for a continued fraction of the second class.)

These conclusions follow by induction from the formulas (3.2.1a) and (3.2.1b) below:

$$A_q - A_{q-1} = b_q A_{q-1} - A_{q-1} + a_q A_{q-2}$$

$$(3.2.1a) \quad A_q - A_{q-1} = (b_q - 1)A_{q-1} + a_q A_{q-2}$$

$$(3.2.1b) \quad B_q - B_{q-1} = (b_q - 1)B_{q-1} + a_q B_{q-2}$$

Corollary 3.2.2

$$\frac{A_q}{A_{q-1}} = b_q + \cfrac{a_q}{b_{q-1} + \cfrac{a_{q-1}}{b_{q-2} + \cdots + \cfrac{a_1}{b_0}}} \, .$$

From equation (3.2.1a) we have

$$\frac{A_q}{A_{q-1}} = b_q + a_q \frac{A_{q-2}}{A_{q-1}} \, ,$$

$$= b_q + a_q \frac{1}{\frac{A_{q-1}}{A_{q-2}}} \, .$$

By (3.2.1)

$$\frac{A_{q-1}}{A_{q-2}} = b_{q-1} + a_{q-1} \frac{A_{q-3}}{A_{q-2}} \, ,$$

$$\therefore \quad \frac{A_q}{A_{q-1}} = b_q + \cfrac{a_q}{b_{q-1} + a_{q-1} \cfrac{1}{\cfrac{A_{q-2}}{A_{q-3}}}} \, ;$$

and

$$\frac{A_{q-2}}{A_{q-3}} = b_{q-2} + a_{q-2}\frac{A_{q-4}}{A_{q-3}} \quad ,$$

$$\therefore \quad \frac{A_q}{A_{q-1}} = b_q + \cfrac{a_q}{b_{q-1} + \cfrac{a_{q-1}}{b_{q-2} + a_{q-2}\cfrac{1}{\cfrac{A_{q-3}}{A_{q-4}}}}} \quad ;$$

and so on, until

$$\frac{A_q}{A_{q-1}} = b_q + \cfrac{a_q}{b_{q-1} + \cfrac{a_{q-1}}{b_{q-2} + \cdot \; \cdot \; \cdot \; \cfrac{a_1}{b_0}}} \quad .$$

And in like manner, by employing equation (3.2.1b), we have

$$\frac{B_q}{B_{q-1}} = b_q + \cfrac{a_q}{b_{q-1} + \cfrac{a_{q-1}}{b_{q-2} + \cdot \; \cdot \; \cdot \; \cfrac{a_2}{b_1}}} \quad .$$

<u>Corollary 3.2.3</u>

$$\frac{A_q}{B_q} - \frac{A_{q-1}}{B_{q-1}} = (-1)^{q-1}\frac{a_1 a_2 \cdots a_q}{B_q B_{q-1}} \quad .$$

This is easily obtained by use of the determinant formula (3.2.2). From this formula we also see that the approximants, as calculated by the recurrence formulas, are not necessarily in their lowest terms.

Corollary 3.2.4

$$\frac{A_q}{B_q} = b_0 + \frac{a_1}{B_0 B_1} - \frac{a_1 a_2}{B_1 B_2} + \frac{a_1 a_2 a_3}{B_2 B_3} - \cdots (-1)^{q-1}\frac{a_1 a_2 \cdots a_q}{B_{q-1} B_q} \quad .$$

Using the relationship

$$\frac{A_q}{B_q} = \frac{A_0}{B_0} + \left(\frac{A_1}{B_1} - \frac{A_0}{B_0}\right) + \left(\frac{A_2}{B_2} - \frac{A_1}{B_1}\right) + \cdots + \left(\frac{A_q}{B_q} - \frac{A_{q-1}}{B_{q-1}}\right)$$

and by (3.2.1) and corollary 3.2.3 we have our result.

Corollary 3.2.5

$$A_q B_{q-2} - A_{q-2} B_q = (-1)^{q-2} b_q a_1 a_2 \cdots a_{q-1} \quad .$$

We have

$$\begin{aligned} A_q B_{q-2} - A_{q-2} B_q &= (b_q A_{q-1} + a_q A_{q-2}) B_{q-2} - A_{q-2}(b_q B_{q-1} + a_q B_{q-2}) \\ &= b_q (A_{q-1} B_{q-2} - A_{q-2} B_{q-1}) \\ &= (-1)^{q-2} b_q a_1 a_2 \cdots a_{q-1} \quad . \end{aligned}$$

Corollary 3.2.6

$$\frac{A_q}{B_q} - \frac{A_{q-2}}{B_{q-2}} = (-1)^{q-2}\frac{b_q a_1 a_2 \cdots a_{q-1}}{B_q B_{q-2}} \quad .$$

This follows from corollary 3.2.5.

Corollary 3.2.7

$$\frac{\dfrac{A_q}{B_q} - \dfrac{A_{q-1}}{B_{q-1}}}{\dfrac{A_{q-1}}{B_{q-1}} - \dfrac{A_{q-2}}{B_{q-2}}} = - \frac{a_q B_{q-2}}{B_q}$$

$$= - \frac{a_q B_{q-2}}{b_q B_{q-1} + a_q B_{q-2}} .$$

This is a consequence of corollary 3.2.3 and formulas (3.2.1).

Corollary 3.2.8

In a continued fraction of the first class, the even approximants form an increasing series, and the odd approximants, a decreasing series; and every even approximant is less than, and every odd approximant greater than, the following approximant.

In a continued fraction of the second class, subject to the restriction $b_q - 1 \geq a_q$, all the approximants are positive, and form an increasing series.

These results follow at once from corollaries 3.2.3 and 3.2.6.

3.3 In chapter I we made reference to the definition of convergence and divergence for non-terminating continued fractions. It was stated that the fraction is said to be convergent if

$$\lim_{n\to\infty} \frac{A_n}{B_n} \tag{3.3.1}$$

is finite and definite, and if not, the fraction is then said to be divergent. The fraction is said to oscillate if the limit (3.3.1) fluctuates between a certain number of values according to the integral value of <u>n</u>. In the next few sections we will deal with the convergence, divergence, and oscillation of continued fractions of the first and second class.[1]

3.4

<u>Theorem 3.4</u>

A continued fraction of the first class cannot be divergent; and it will be convergent or oscillating if any one of the complete quotients $x_1, x_2, \ldots, x_n, \ldots$ converge or oscillate.

The latter part of the theorem is obtained from the equation

$$x = b + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots \cfrac{a_n}{x_n}}} .$$

1. Chrystal, <u>Algebra</u>, Volume II, Pages 477-484

By corollary 3.2.8, the even approximants continually increase and the odd approximants continually decrease, while any odd approximant is greater than any following even approximant. It follows that

$$\lim_{n\to\infty}\frac{A_{2n}}{B_{2n}}=\alpha\,,\quad \lim_{n\to\infty}\frac{A_{2n-1}}{B_{2n-1}}=\beta\quad,$$

where α and β are two finite quantities, and $\beta\geq\alpha$. If $\beta=\alpha$, the fraction is convergent; if $\beta>\alpha$, it oscillates; and no other case can arise.

3.5

Theorem 3.5

A continued fraction of the first class is convergent if the series

$$\sum_{n=2}^{\infty} b_{n-1}\frac{b_n}{a_n}$$

is divergent.

Since all the quantities involved are positive, we have

$$B_n = b_n B_{n-1} + a_n B_{n-2}\quad ;$$

$$B_{n-1} = b_{n-1} B_{n-2} + a_{n-1} B_{n-3},\quad B_{n-1} > b_{n-1} B_{n-2}\ ;$$

$$B_{n-2} = b_{n-2} B_{n-3} + a_{n-2} B_{n-4},\quad B_{n-2} > b_{n-2} B_{n-3}\ ;$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$B_4 = b_4 B_3 + a_4 B_2 \;, \quad B_4 > b_4 B_3 \;;$$

$$B_3 = b_3 B_2 + a_3 B_1 \;, \quad B_3 > b_3 B_2 \;;$$

$$B_2 = b_2 B_1 + a_2 B_0 \;, \quad B_2 > b_2 B_1 \;;$$

$$B_1 = b_1 B_0 \,.$$

Hence

$$B_n > (b_n b_{n-1} + a_n) B_{n-2} \,,$$

$$B_{n-1} > (b_{n-1} b_{n-2} + a_{n-1}) B_{n-3} \,,$$

$$B_{n-2} > (b_{n-2} B_{n-3} + a_{n-2}) B_{n-4} \,,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$B_4 > (b_4 b_3 + a_4) B_2 \;,$$

$$B_3 > (b_3 b_2 + a_3) B_1 \;,$$

$$B_2 = (b_2 b_1 + a_2) B_0 \,.$$

Therefore

$$B_n B_{n-1} > B_0 B_1 (a_2 + b_1 b_2)(a_3 + b_2 b_3) \cdots (a_n + b_{n-1} b_n) \;,$$

and, since $B_0 = 1$, $B_1 = b_1$, we have

$$\frac{B_n B_{n-1}}{a_1 a_2 \cdots a_n} > \frac{b_1}{a_1} \left(1 + \frac{b_1 b_2}{a_2}\right)\left(1 + \frac{b_2 b_3}{a_3}\right) \cdots \left(1 + \frac{b_{n-1} b_n}{a_n}\right) \,.$$

Now, since $\sum_{n=2}^{\infty} b_{n-1} \frac{b_n}{a_n}$ is divergent, by applying the theory of infinite products (see Chrystal, Algebra, Volume 2, pages 135-137), we have that

$$(3.5.1) \qquad \prod_{n=2}^{\infty} \left(1 + b_{n-1} \frac{b_n}{a_n}\right)$$

diverges to $+\infty$, and therefore,

$$\lim_{n\to\infty} \frac{B_n B_{n-1}}{a_1 a_2 \cdots a_n} = +\infty \ .$$

Hence,

$$\lim_{n\to\infty} \left(\frac{A_{2n}}{B_{2n}} - \frac{A_{2n-1}}{B_{2n-1}} \right) = \lim_{n\to\infty} \frac{a_1 a_2 \cdots a_{2n}}{B_{2n} B_{2n-1}} = 0 \ ,$$

that is, the continued fraction converges.

Corollary 3.5.1

The fraction in question is convergent if

$$\lim_{n\to\infty} b_{n-1} \frac{b_n}{a_n} > 0 \quad .$$

By the theory of infinite products, the product (3.5.1) will diverge if

$$\lim_{n\to\infty} b_{n-1} \frac{b_n}{a_n} > 0 \ ,$$

and, therefore, the continued fraction will converge.

Corollary 3.5.2

If $\lim_{n\to\infty} \frac{b_n}{a_n} > 0$ and $\sum_{n=2}^{\infty} b_n$ is divergent, the fraction in question converges.

For then $\sum_{n=2}^{\infty} b_{n-1} \frac{b_n}{a_n}$ is divergent, and theorem 3.5 applies.

3.6

Theorem 3.6

If a continued fraction of the first class is reduced to the form

(3.6.1)
$$b_0 + \cfrac{1}{e_1 + \cfrac{1}{e_2 + \cfrac{1}{e_3 + \cdots \cfrac{1}{e_n + \cdots}}}}$$

by the equivalence transformation section 1.10, then it is convergent if at least one of the series

(3.6.2) $e_1 + e_3 + e_5 + \ldots$

(3.6.3) $e_2 + e_4 + e_6 + \ldots$

is divergent, oscillating if both these series are convergent.

We have

$$e_1 = \frac{b_1}{a_1},\ e_2 = \frac{a_1 b_2}{a_2},\ e_3 = \frac{a_2 b_3}{a_1 a_3},\ \ldots,$$

$$e_n = \frac{b_n a_{n-1} a_{n-3} \cdots}{a_n a_{n-2} \cdots} .$$

By the theory of continuants (see Chrystal, Algebra, Volume 2, pages 466-468), we have

(3.6.4) $0 < B_n < (1+e_1)(1+e_2)(1+e_3)\cdots(1+e_n)$.

We shall also show that

$$(3.6.5) \qquad B_{2n-1} \geq e_1 + e_3 + \cdots + e_{2n-1} \geq e_1$$

and

$$(3.6.6) \qquad B_{2n} \geq e_2 + e_4 + \cdots + e_{2n} \geq 1 \quad .$$

We have, by corollary 2.3.2,

$$\frac{A_{2n-1}}{B_{2n-1}} - \frac{A_{2n}}{B_{2n}} = \frac{1}{B_{2n} B_{2n-1}} \quad .$$

If we let $e_1 \neq 0$, then

$$B_1 = e_1 B_0 + B_{-1} = e_1 \;,$$

$$B_2 = e_2 B_1 + B_0 \geq 1$$

$$B_3 = e_3 B_2 + B_1 \geq e_1$$

$$B_4 = e_4 B_3 + B_2 \geq 1$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$B_{2n-1} = e_{2n-1} B_{2n-2} + B_{2n-3} \geq e_1$$

$$B_{2n} = e_{2n} B_{2n-1} + B_{2n-2} \geq 1$$

and we see that neither B_{2n} nor B_{2n-1} can vanish and, also, that (3.6.5) and (3.6.6) hold. Hence, if both

$$\lim_{n\to\infty} B_{2n} \quad \text{and} \quad \lim_{n\to\infty} B_{2n-1}$$

are finite, the fraction (3.6.1) will oscillate. And if one of them is infinite, the fraction will converge.

If both the series (3.6.2) and (3.6.3) converge, the series

$$e_1 + e_2 + e_3 + \cdots + e_n$$

will converge; and the product

$$\prod_{n=1}^{\infty} (1+e_n)$$

will be finite (theory of infinite products), and we then have B_n bounded by (3.6.4). In this case, both B_{2n} and B_{2n-1} will be finite and the continued fraction will oscillate.

If the series (3.6.2) is divergent, then by (3.6.5)

$$\lim_{n\to\infty} B_{2n-1} = \infty$$

and the fraction is convergent. Also, if the series (3.6.3) diverges, then by (3.6.6)

$$\lim_{n\to\infty} B_{2n} = \infty$$

and the fraction converges.

Theorem 3.6A

If all the elements of the continued fraction

$$b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cdots}}$$

are positive, a necessary and sufficient condition that the fraction converges is that the series $\sum_{n=0}^{\infty} b_n$ diverges.

<u>Theorem 3.6B</u>

A necessary conditon for the convergence of the continued fraction

$$b_0+\cfrac{1}{b_1+\cfrac{1}{b_2+\cdots}}$$

where b_0 , ... , b_n are real or complex elements, is that the series $\sum_{n=0}^{\infty}|b_n|$ diverges.

3.7

<u>Theorem 3.7</u>

If an infinite continued fraction of the second class of the form

(3.7.1) $$L = \cfrac{a_1}{b_1-\cfrac{a_2}{b_2-\cfrac{a_3}{b_3-\cdots\cfrac{a_n}{b_n-\cdots}}}}$$

is such that

(3.7.2) $$b_n \geq a_n + 1$$

for all values of <u>n</u>, it converges to a finite limit L not greater than unity.

(1.) If the sign $>$ occurs in the condition (3.7.2) then $L < 1$.

(2.) If the sign $=$ occurs alone, then $L = 1 - \frac{1}{S}$, where

(3.7.3) $\quad S = 1 + a_1 + a_1 a_2 + a_1 a_2 a_3 + \cdots + a_1 a_2 \cdots a_n + \cdots,$

so that $L = 1$ if series (3.7.3) diverges and $L < 1$ if the series converges.

These results may be obtained by applying certain characteristic properties of the fraction (3.7.1). We will first state these properties, prove them, and then use them to prove theorem 3.7.

(3.7.4) $\quad A_n - A_{n-1} \geq a_1 a_2 \cdots a_n$

(3.7.5) $\quad A_n \geq a_1 + a_1 a_2 + a_1 a_2 a_3 + \cdots + a_1 a_2 \cdots a_n$

(3.7.6) $\quad B_n - B_{n-1} \geq a_1 a_2 \cdots a_n$

(3.7.7) $\quad B_n \geq 1 + a_1 + a_1 a_2 + \cdots + a_1 a_2 \cdots a_n$

(3.7.8) $\quad B_n - A_n \geq 1$

To prove (3.7.4) we observe that

$$A_n - A_{n-1} = (b_n - 1)A_{n-1} - a_n A_{n-2} \quad .$$

And, since A_n, B_n are positive and increase with $\underline{n}$ by corollary 3.2.1, we have

$$A_n - A_{n-1} \geq a_n(A_{n-1} - A_{n-2}) \quad \text{as } b_n \geq a_n + 1 \;;$$

$$A_{n-1} - A_{n-2} \geq a_{n-1}(A_{n-2} - A_{n-3}) \text{ as } b_{n-1} \geq a_{n-1} + 1 \;;$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$A_2 - A_1 \geq a_2(A_1 - A_0) \qquad \text{as } b_2 \geq a_2 + 1 \;;$$

$$A_1 - A_0 = a_1 \qquad \text{since } A_0 = 0 \quad .$$

Therefore

$$A_n - A_{n-1} \geqq a_1 a_2 \cdots a_n$$

where the upper sign must be taken if it occurs anywhere in the conditions.

To prove (3.7.5), we have

$$A_n \geqq A_{n-1} + a_1 a_2 \cdots a_n$$

$$A_{n-1} \geqq A_{n-2} + a_1 a_2 \cdots a_{n-1}$$

$$A_{n-2} \geqq A_{n-3} + a_1 a_2 \cdots a_{n-2}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$A_2 \geqq A_1 + a_1 a_2$$

$$A_1 = a_1$$

and by substituting we obtain the required result

$$A_n \geqq a_1 + a_1 a_2 + \cdots + a_1 a_2 \cdots a_n \ .$$

And in like manner, we can verify (3.7.6) and (3.7.7) with $B_0 = 1$.

To establish (3.7.8), we have

$$B_n - A_n \geqq (1 + a_1 + \cdots + a_1 a_2 \cdots a_n) - (a_1 + \cdots + a_1 a_2 \cdots a_n),$$

$$B_n - A_n \geqq 1 \ .$$

Now, since the approximants of a continued fraction of the second class form an increasing series of positive quantities (corollary 3.2.8), there can be no oscillation for (3.7.1). And from (3.7.8) it follows that

$$B_n - A_n \geqq 1 \ , \quad B_n \geqq 1 + A_n \ ,$$

$$\frac{A_n}{B_n} \leqq 1 - \frac{1}{B_n} \ .$$

Therefore, since $B_n > 1$, we have $\underline{L}$ converging to a finite limit ≤ 1. If the sign $>$ occurs in the condition (3.7.2), $L < 1$. If the sign $=$ occurs alone, we have

$$\lim_{n\to\infty} \frac{A_n}{B_n} = 1 - \lim_{n\to\infty} \frac{1}{B_n} = 1 - \frac{1}{S} \quad .$$

Hence, if $\underline{S}$ converges, $L < 1$; if it diverges, $L = 1$.

3.8

We now offer a few theorems without proof.

Theorem 3.8.1[1]

The continued fraction

$$1 + \cfrac{a}{1 + \cfrac{a}{1 + \cdots}}$$

converges excepting when $a = -\frac{1}{4} - c$, where $\underline{c}$ is real and positive. When convergent, its value is $1/2$ if $a = -1/4$, and is equal to the one of the quantities

$$\frac{1 \pm (1+4a)^{1/2}}{2}$$

having the larger modulus if $a \neq 1/4$.

1. Wall, Continued Fractions, Page 39.

In chapter IV, we will show that

$$1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}$$

is equal to $(1+\sqrt{5})/2$.

<u>Theorem 3.8.2</u>[1]

The continued fraction

$$\cfrac{1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\cdots}}}$$

converges for

$$|a_n| \leq 1/4 \ , \quad n = 2, 3, \ldots \quad .$$

The following theorems sum up the more important convergence theorems.

<u>Theorem 3.8.3</u>[2]

If in

$$\cfrac{1}{b_1+\cfrac{1}{b_2+\cdots}} \qquad (b > 0) \quad ,$$

the series $\sum_{n=1}^{\infty} b_n$ diverges the continued fraction converges, while it diverges if $\sum_{n=1}^{\infty} b_n$ is convergent.

1. Wall, <u>Continued Fractions</u>, Page 63.
2. Wall, <u>Continued Fractions</u>, Chapter II.

Theorem 3.8.4 [1]

If in

$$\cfrac{1}{a_1+ib_1+\cfrac{1}{a_2+ib_2+\cfrac{1}{a_3+ib_3+\cdots}}}$$

the a_q elements have all the same sign and the b_q are alternately positive and negative (zero values are permissible for either a_q or b_q), the continued fraction will converge if

$$\sum_{n=0}^{\infty} |a_n+ib_n|$$

is divergent. On the other hand, if

$$\sum_{n=0}^{\infty} |a_n+ib_n|$$

is convergent and either a_q or b_q fulfill the condition just stated concerning their signs, the even and odd approximants have separate limits.

3.9 Irrationality of Certain Continued Fractions.[2]

If

$$a_1, a_2, \ldots, a_n, \quad b_1, b_2, \ldots, b_n$$

are all positive integers then

1. Van Vleck, Transactions of the American Mathematical Society, Volume 2, pages 215-233
2. Chrystal, Algebra, Volume II, pages 484-6.

Theorem 3.9.1

The infinite continued fraction

(3.9.1)
$$\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cdots\cfrac{a_n}{b_n+\cdots}}}$$

converges to an irrational limit provided that after some finite value of n the condition $a_n \geq b_n$ is always satisfied.

Theorem 3.9.2

The infinite continued fraction

(3.9.2)
$$\cfrac{a_1}{b_1-\cfrac{a_2}{b_2-\cdots\cfrac{a_n}{b_n-\cdots}}}$$

converges to an irrational limit provided that after some finite value of n the condition $b_n \geq a_n+1$ is always satisfied, where the sign $>$ need not always occur but must occur infinitely often.

To prove theorem (3.9.2), let us first suppose that the condition $b_n \geq a_n+1$ holds from the first. Then (3.9.2) converges, by theorem 3.7, to a positive value less than one. Let us assume that it converges to a rational limit c_2/c_1, where c_1, c_2 are positive integers and $c_1 > c_2$.

Let

$$d_3 = \cfrac{a_2}{b_2 - \cfrac{a_3}{b_3 - \cdots}} \quad .$$

Since the sign $>$ must occur among the conditions

$$b_2 \geq a_2+1, \quad b_3 \geq a_3+1, \quad \ldots ,$$

d_3 must be a positive quantity less than one. Now, by our assumptions,

$$\frac{c_2}{c_1} = \frac{a_2}{b_2 - d_3} \quad ,$$

and, therefore,

$$d_3 = \frac{b_2 c_2 - a_2 c_1}{c_2} = \frac{c_3}{c_2}$$

where $c_3 = b_2 c_2 - a_2 c_1$ is an integer which must be positive and less than c_2. Then, as before, we have $d_4 = c_4/c_3$, where c_4 is a positive integer less than 1.

Since the sign $>$ occurs infinitely often among the conditions

$$b_n \geq a_n + 1,$$

this process can be repeated as often as we please. The hypothesis that the fraction (3.9.2) is rational therefore requires the existence of an infinite number of positive integers

$$c_1, c_2, c_3, c_4, \ldots ,$$

such that

$$c_1 > c_2 > c_3 > c_4 \ldots;$$

but this is impossible since c_1 is finite. Hence (3.9.2) is irrational.

Next, suppose the condtion

$$b_n \geq a_n + 1$$

holds after n=m; then by what has been shown above,

$$y = \cfrac{a_{m+1}}{b_{m+1} - \cfrac{a_{m+2}}{b_{m+2} - \cdots}}$$

is irrational.

We now have

$$Y = \cfrac{a_1}{b_1 - \cfrac{a_2}{b_2 - \cdots \cfrac{a_m}{b_m - y}}} ,$$

then

$$Y = \frac{(b_m - y)A_{m-1} - a_m A_{m-2}}{(b_m - y)B_{m-1} - a_m B_{m-2}}$$

(3.9.3) $$= \frac{A_m - y\ A_{m-1}}{B_m - y\ B_{m-1}} \quad ,$$

where A_m/B_m and A_{m-1}/B_{m-1} are the ultimate and penultimate approximants of

$$\cfrac{a_1}{b_1 - \cfrac{a_2}{b_2 - \cdots \cfrac{a_m}{b_m}}} \quad .$$

From (3.9.3), we have

(3.9.4) $$y(YB_{m-1} - A_{m-1}) = YB_m - A_m \quad .$$

Now $(YB_{m-1} - A_{m-1})$ and $(YB_m - A_m)$ cannot both be zero for that would involve the equality

$$\frac{A_m}{B_m} = \frac{A_{m-1}}{B_{m-1}}$$

which is inconsistent with corollary (3.2.3). And, if $\underline{Y}$ were rational, (3.9.4) would give a rational value for the irrational $\underline{y}$. $\underline{Y}$ must therefore be irrational.

The proof of theorem (3.9.1) is exactly similar, since the condition $b_n \geq a_n$ requires each of the complete quotients to be positive and less than one.

These two theorems do not include all cases of irrational limits for infinite continued fractions. We have Lord Brouncker's fraction for $\pi/4$ violating the condition of theorem (3.9.1).

CHAPTER IV

Continued Fractions and the Theory of Equations

Since the theory of continued fractions yields an infinite process which can be used to approximate many irrational numbers, there would seem to be a place for continued fractions in the numerical solution of algebraic equations. Although continued fractions can be applied to the solutions of a few special equations, they do not seem to be readily adaptable to the solution of the general algebraic equation

$$x^n + a_1x^{n-1} + \dots + a_n = 0.$$

However, there is a very unique theorem which utilizes finite simple continued fractions in determining the number of, and in the separation of the real roots of an equation. It is the theorem of Vincent which was first published in 1836 in the first volume of Liouville's Journal and more recently included in the Theory of Equations by J. V. Uspensky. The quotations below are from Uspensky, pages 128-132.

4.1

Theorem 4.1 Theorem of Vincent

"Let a,b,c, ... be an arbitrary sequence of positive integers. Transforming an equation without multiple roots by a series of successive substitutions

$x = a + 1/y,\ y = b + 1/z,\ z = c + 1/t$, etc.,

after a number of such substitutions independent of the choice of integers $a, b, c, \ldots$, we come to a transformed equation with not more than one variation."

The proof of this theorem which is based on the theory of continued fractions is found in Uspensky, appendix II. Uspensky also offers a method which overcomes the condition of no multiple roots.

"We will, at first, determine the exact number of positive roots by the following method:

"We notice that the positive roots are either >1 or <1 (excluding the case when 1 is a root). The positive roots >1 may be written in the form $x = 1 + y$ with $y > 0$, while those <1 may be written in the form $x = 1/(1+y)$ where again $y > 0$. The proposed equation is transformed by the substitutions $x = 1 + y$ and $x = 1/(1+y)$ into two easily obtainable equations. If the transformed equations have no variations of sign or just one, the question is settled. For if the equation obtained by the transformation $x = 1 + y$ has no variations, it means that the original equation has no roots >1; and the presence of just one variation in the transformed equation indicates just one root >1 of the proposed equation. Similar conclusions hold for the equation resulting from the substitution $x = 1/(1+y)$.

"If one or both of the transformed equations have more than one variation, we transform them again by the substitutions $y = 1 + z$, $y = 1/(1+z)$, and if necessary continue the transformations by substitutions of the same type until the transformed equations obtained by this process have no more than one variation. The transformed equations with one variation will have a positive root between 0 and ∞. Substituting these values in the

resulting continued fraction, we obtain values for x between which lie the positive real roots of the original equation. This process is best illustrated by examples.

Example 1

Separate the roots of the equation

$$x^3 - 7x + 7 = 0.$$

"Since there are no rational roots, the positive roots are either >1 or <1. The roots >1 are of the form $x = 1 + y$, and those <1 are of the form $x = 1/(1+y)$ with $y > 0$ in both cases. Now, to find the number of roots >1, we transform the original equation by the substitution $x = 1 + y$ and seek the number of positive roots of the transformed equation. Only additions are required to effect this transformation. In our example, the necessary operations are as follows:[1]

1	0	-7	7
1	1	-6	1
1	2	-4	
1	3		
1			

so the transformed equation is

$$y^3 + 3y^2 - 4y + 1 = 0.$$

To perform the transformation

$$x = \frac{1}{1+y}$$

we use two steps. First, x is replaced by $1/x$ which leads to

$$7x^3 - 7x^2 + 1 = 0.$$

1. See Dickson, New First Course in Theory of Equations, page 90.

The effect of this preliminary transformation is the reversal of the order of the coefficients. Next, we set x = 1 + y in the new equation and perform the operation as indicated:

$$
\begin{array}{cccc}
\underline{7} & \underline{-7} & \underline{0} & \underline{1} \\
7 & 0 & 0 & \underline{1} \\
7 & 7 & \underline{7} & \\
7 & \underline{14} & & \\
\underline{7} & & &
\end{array}
\quad .
$$

The final transformed equation is

$$7y^3 + 14y^2 + 7y + 1 = 0 .$$

Instead of replacing x by 1/x and then by 1 + y, we can proceed directly, starting additions from the right and going upward as shown:

$$
\begin{array}{cccc}
 & & & \underline{7} \\
 & & \underline{14} & 7 \\
 & \underline{7} & 7 & 7 \\
\underline{1} & 0 & 0 & 7 \\
\hline
1 & 0 & -7 & 7
\end{array}
\quad .
$$

The underlined numbers, when read down, supply the coefficients 7, 14, 7, 1 of the transformed equation by means of the substitution x = 1/(1+y). Both transformations x = 1 + y and x = 1/(1+y) can be performed in the same scheme, shown below with the parallelogrammatic arrangement of numbers:

$$\begin{array}{cccc}
 & & & \underline{7} \\
 & & \underline{14} & 7 \\
 & \underline{7} & 7 & 7 \\
\underline{1} & 0 & 0 & 7 \\
\hline
1 & 0 & -7 & 7 \\
\hline
1 & 0 & -6 & \underline{1} \\
1 & 2 & \underline{-4} & \\
1 & \underline{3} & & \\
\underline{1} & & &
\end{array}$$

Since the equation

$$7y^3 + 14y^2 + 7y + 1 = 0$$

has no variations, it has no positive roots, and hence the proposed equation has no roots of the form $x = 1/(1+y)$ with $y > 0$, that is, no root between 0 and 1. But the equation

$$y^3 + 3y^2 - 4y + 1 = 0$$

resulting from the substitution $x = 1 + y$ has two variations, and we have to treat it further by making the two substitutions

$$y = 1 + z \quad \text{and} \quad y = 1/(1+z).$$

"The necessary calculations are shown in the scheme:

$$
\begin{array}{cccc}
 & & & \underline{1} \\
 & & \underline{-1} & 1 \\
 & \underline{-2} & -2 & 1 \\
\underline{1} & 0 & -3 & 1 \\
\hline
1 & 3 & -4 & 1 \\
\hline
1 & 4 & 0 & \underline{1} \\
1 & 5 & \underline{5} & \\
1 & \underline{6} & & \\
1 & & &
\end{array}
$$

The equation resulting from the substitution $y = 1 + z$ is

$$z^3 + 6z^2 + 5z + 1 = 0$$

has no variations and no positive roots, but the equation resulting from the substitution $y = 1/(1+z)$

$$z^3 - z^2 - 2z + 1 = 0$$

still has two variations and must be subjected again to the transformations

$$z = 1 + t \quad \text{and} \quad z = 1/(1+t) \ .$$

The necessary calculations are

$$
\begin{array}{cccc}
 & & & \underline{1} \\
 & & \underline{1} & 1 \\
 & \underline{-2} & 0 & 1 \\
\underline{-1} & -2 & -1 & 1 \\
\hline
1 & -1 & -2 & 1 \\
\hline
1 & 0 & -2 & \underline{-1} \\
1 & 1 & \underline{-1} & \\
1 & 2 & & \\
\underline{1} & & &
\end{array}
$$

so that the transformed equations are

$$t^3 + 2t^2 - t - 1 = 0$$

and

$$t^3 + t^2 - 2t - 1 = 0$$

and have each only one variation and hence only one positive root which lies in the interval $\underline{0}$ to ∞. Now the first equation results from the original one by the substitutions

$$x = 1 + y, \quad y = \frac{1}{1+z}, \quad z = 1 + t,$$

which can be combined into one:

$$x = 1 + \frac{1}{2+t} \quad .$$

The substitutions leading to the second equation

$$x = 1 + y, \quad y = \frac{1}{1+z}, \quad z = \frac{1}{1+t}$$

can be combined into

$$x = 1+\cfrac{1}{1+\cfrac{1}{1+t}} \qquad .$$

Each of thetransformed equations having just one positive root, there are two positive roots to the proposed equation, and the intervals within which they lie are obtained by taking the extreme values $\underline{t} = 0$ and $\underline{t} = \infty$ in the formulas expressing $\underline{x}$. Thus, we find two intervals

$$(1, 3/2) \quad \text{and} \quad (3/2, 2)$$

each containing one root of the equation

$$x^3 - 7x + 7 = 0.$$

"The successive substitutions that serve to pass from $\underline{x}$ to $\underline{t}$ are immediately seen if the results of the transformations applied are arranged in a scheme resembling a genealogical tree:

$x^3 - 7x + 7$

$x = 1/(1+y)$ | $x = 1+y$

$7y^3 + 14y^2 + 7y + 1$ (no var.) | $y^3 + 3y^2 - 4y + 1$ (2 var.)

$y = 1/(1+z)$ | $y = 1+z$

$z^3 - z^2 - 2z + 1$ (2 var.) | $z^3 + 6z^2 + 5z + 1$ (no var.)

$z = 1/(1+t)$ | $z = 1+t$

$t^3 + t^2 - 2t - 1$ (1 var.) | $t^3 + 2t^2 - t - 1$ (1 var.)

"To find the number of negative roots, we substitute $-\underline{x}$ for $\underline{x}$ in our original equation to obtain the transformed equation

$$x^3 - 7x - 7 = 0$$

which has only one variation and therefore one positive root which, by inserting $x = 1, 2, 3, \ldots$ and observing the signs of the results, is found to be contained between 3 and 4. Hence, the proposed equation has one negative root in the interval $(-4, -3)$."

We now introduce a method which enables us to approximate these roots by the use of continued fractions.[1] This method was proposed by Lagrange and at that time was dependent on Sturm's process of root separation. The method, which is more easily explained by examples, becomes cumbersome when applied to equations of a high degree.

As we have shown, the equation

$$f(x) = x^3 - 7x + 7 = 0$$

has roots in the intervals $(1, 3/2)$, $(3/2, 2)$ and $(-4, -3)$.

To approximate the roots between $(1, 3/2)$ and $(3/2, 2)$, let $x = 1 + 1/y$ where $y > 1$.

Since the Taylor series expansion

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + \ldots + f^{(n)}(x)\frac{h^n}{n!} = 0$$

is finite when $f(x)$ is a polynomial ($\underline{n}$ is degree of $f(x)$), we can obtain the transformed equation in $\underline{y}$

1. DeMorgan, Mathematics, Volume I, Theory of Algebraical Equations, pages 127-8, 132-6.

with the aid of the successive derivatives of f(x).

We have

$$f'(x) = 3x^2 - 7$$
$$f''(x) = 6x$$
$$f'''(x) = 6$$

and hence,

$$f(1+\frac{1}{y}) = f(1) + f'(1)\frac{1}{y} + f''(1)\frac{1}{2!y^2} + f'''(1)\frac{1}{3!y^3} = 0$$
$$= 1 - \frac{4}{y} + \frac{3}{y^2} + \frac{1}{y^3} = 0 .$$

Therefore,

$$f_1 = y^3 - 4y^2 + 3y + 1 = 0$$

is the transformed equation.

By substituting the numbers 1,2,3,... in f_1 and noticing the changes of sign, we see that f_1 has two positive roots in the intervals (1,2) and (2,3). We now have to set up two transformations

$$y = 1 + \frac{1}{z} \text{ and } y = 2 + \frac{1}{z'}, \qquad z, z' > 1$$

corresponding respectively tothe larger and smaller positive roots of f(x).

And we have

$$f'(x) = 3y^2 - 8y + 3$$
$$f''(x) = 6y - 8$$
$$f'''(x) = 6$$

and the transformed equations in $\underline{z}$ and $\underline{z}'$ are

$$f_2 = z^3 - 2z^2 - z + 1 \qquad (0,1)\ (2,3)$$

$$f_3 = z'^3 + z'^2 - 2z - 1 \qquad (1,2) \quad .$$

By observing when f_2 and f_3 change signs, we find that f_2 has two positive roots between (0,1) and (2,3) and f_3 has one positive root between (1,2). We can disregard the root of f_2 between (0,1) since $\underline{z}$ must be greater than $\underline{1}$.

We must now perform the transformations

$$z = 2 + \frac{1}{u} \text{ and } z' = 1 + \frac{1}{u'} \qquad u, u' > 1$$

and obtain the equations

$$f_4 = u^3 - 3u^2 - 8u - 6$$

$$f_5 = u'^3 - 3u'^2 - 8u' - 6 \quad .$$

We notice that the coefficients of f_4 and f_5 are the same so we now need only to consider the equation

$$f_4 = u^3 - 3u^2 - 8u - 6$$

which has only one positive root in the interval (4,5). The next transformation for both roots is

$$u = u' = 4 + \frac{1}{v} \qquad v > 1.$$

Concluding the transformations at this step, we have for the positive roots of f(x)

$$x_1 = 1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{4+\cdots}}} = \frac{22}{13} = 1.69^{+} ,$$

where

$$x_1 - \frac{22}{13} < \frac{1}{13^2} \quad ;$$

$$x_2 = 1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{4+\cdots}}} = \frac{19}{14} = 1.35^{+}$$

where

$$x_2 - \frac{19}{14} < \frac{1}{14^2} \quad ;$$

we likewise can approximate the negative root of $f(x)$ which lies in the interval $(-3,-4)$ and this turns out to be

$$x_3 = -3-\cfrac{1}{20+\cfrac{1}{3+\cdots}} = -\frac{186}{61} = -3.049^{+} ,$$

where

$$-\frac{186}{61} - x_3 < \frac{1}{61^2} \quad .$$

We have then solved the equation $x^3-7x+7=0$ completely by the use of continued fractions.

The accuracy of Lagrange's method is illustrated by the following example:

Example

Approximate thepositive root of the equation

$$x^3 - 2x - 5 = 0$$

by continued fractions and by Newton's method.

Omitting the necessary calculations, we have

$$x = 2+\cfrac{1}{10+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{12+ \ddots}}}}}}}}}$$

and

$$\frac{A_q}{B_q} = \frac{16415}{7837} = 2.0945514865 ,$$

$$\frac{16415}{7837} - x < \frac{1}{(7837)^2} = .0000000163 .$$

By Newton's method, the positive root was found to be

$$2.0945514815 ,$$

the error being only

$$0.0000000051 .$$

4.2

As we have seen in Chapter II, continued fractions and quadratic equations are rather intimately connected. We will further this relationship by proving and demonstrating various theorems and methods concerning the solution of the general quadratic equation by continued fractions.

<u>Theorem 4.2.1</u>

If a quadratic equation

$$ax^2 + bx + c = 0$$

with rational coefficients has irrational roots and

I. One variation of sign

or

II. Two variations of sign such that

$$|b| - 1 > ac$$

or

III. No variations of sign such that

$$|b| - 1 > ac$$

the roots of the equation are

$$x_1 = -c\left(\cfrac{1}{b - \cfrac{ac}{b - \cfrac{ac}{b - \ddots}}}\right)$$

and

$$x_2 = -\frac{1}{a}\left(b - \cfrac{ac}{b - \cfrac{ac}{b - \ddots}}\right) .$$

In line with this theorem, we use the following lemma which can be proved by induction:

<u>Lemma 4.2.1</u>

If

$$x_1 = \frac{A_n}{B_n} = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}} \qquad \text{as } n \to \infty$$

and

$$x_2 = \frac{A_n'}{B_n'} = -b_0 + \cfrac{a_1}{-b_1 + \cfrac{a_2}{-b_2 + \ddots}} \qquad \text{as } n \to \infty ,$$

with $a_n, b_n > 0$, then

$$x_1 = -x_2 .$$

From the general quadratic equation, we have:

$$ax^2 + bx + c = 0 \qquad a, b, c \neq 0$$

$$ax^2 + bx = -c$$

$$x(ax+b) = -c$$

$$x = -\frac{c}{b+ax}$$

(4.2.1)
$$x_1 = -c\left(\cfrac{1}{b-\cfrac{ac}{b-\cfrac{ac}{b-\ddots}}}\right) ;$$

and

$$ax^2 + bx + c = 0$$

$$ax^2 = -bx - c$$

$$ax = -b - \frac{c}{x} = -b - \frac{ac}{ax}$$

$$ax = -b-\cfrac{ac}{-b-\cfrac{ac}{-b-\ddots}}$$

$$x_2 = \frac{1}{a}\left(-b-\cfrac{ac}{-b-\cfrac{ac}{-b-\ddots}}\right) .$$

By lemma 4.2.1, we have

(4.2.2) $$x_2 = -\frac{1}{a}\left(b - \cfrac{ac}{b - \cfrac{ac}{b - \ddots}}\right) .$$

Letting

$$\alpha = b - \cfrac{ac}{b - \cfrac{ac}{b - \ddots}}$$

(4.2.1) becomes

(4.2.3) $$x_1 = -\frac{c}{\alpha}$$

and (4.2.2) becomes

(4.2.4) $$x_2 = -\frac{\alpha}{a} = \frac{1}{a}\left(-b + \frac{ac}{\alpha}\right) .$$

For case I, we must have either a or c negative, but not both. In either case, if b is not positive it can be made positive. We shall require $b > 0$ to shorten the proof.

We first need to show that α is finite and definite and hence, x_1 and x_2 are also finite and definite. This can be accomplished by proving α converges for all possible values of a, b, and c. Since ac is negative, α will be a fraction of the first class.

By corollary 3.5.1, $\underline{\alpha}$ will converge if

$$\lim_{n\to\infty} b_{n-1}\frac{b_n}{a_n} > 0$$

and for $\underline{\alpha}$

$$b_{n-1} = b_n = b,$$

$$a_n = -(ac) = |ac|,$$

$$\lim_{n\to\infty} \frac{b^2}{|ac|} > 0$$

and, therefore, $\underline{\alpha}$ converges.

Now, if $\underline{x_1}$ and $\underline{x_2}$ are the desired roots, the equation formed with $\underline{x_1}$ and $\underline{x_2}$ as roots must necessarily have the same coefficients as the original equation. To demonstrate this, we have by (4.2.3) and (4.2.4)

(4.2.5) $$x_1 x_2 = \frac{c}{a}$$

or

$$x_1 x_2 = -\frac{c}{\alpha}\left[\frac{1}{a}\left(-b + \frac{ac}{\alpha}\right)\right]$$

(4.2.6) $$= \frac{bc}{a\alpha} - \frac{c^2}{\alpha^2} .$$

We next form the equation in $\underline{x}$ with $\underline{x_1}$ and $\underline{x_2}$ as roots:

$$\left(x + \frac{c}{\alpha}\right)\left(x + \frac{b}{a} - \frac{c}{\alpha}\right) = 0$$

$$x^2 + \frac{b}{a}x - \frac{c}{\alpha}x + \frac{c}{\alpha}x + \frac{bc}{a\alpha} - \frac{c^2}{\alpha^2} = 0$$

$$x^2 + \frac{b}{a}x + \frac{bc}{a\alpha} - \frac{c^2}{\alpha^2} = 0$$

and by (4.2.5) and (4.2.6)

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$ax^2 + bx + c = 0 \quad .$$

Hence, case I is proved.

For case II, we first show by induction, as in lemma 4.2.1 that if

$$x_1 = b_0 - \cfrac{a_1}{b_1 - \cfrac{a_2}{b_2 - \cdots}}$$

and

$$x_2 = -b_0 - \cfrac{a_1}{-b_1 - \cfrac{a_2}{-b_2 - \cdots}}$$

then

$$x_1 = -x_2 \quad .$$

Again, by the conditions of case II, we can always make $b > 0$. Here, since $\underline{ac}$ is positive, $\underline{\alpha}$ is a fraction of the second class and will converge, by theorem 3.7, if

$$|b| - 1 > ac \quad .$$

Now, by applying the theory of case I, we have $\underline{x}_1$ and $\underline{x}_2$ the roots for case II.

Case III is proven exactly as case II. Cases II and III will also hold for

$$|b| - 1 = ac$$

when

$$b^2 = a^2c^2 + 2ac + 1 > 4ac.$$

We have

$$b^2 - 4ac = a^2c^2 - 2ac + 1 = (ac-1)^2$$

or that the roots of the equation are rational numbers. This is checked by theorem 3.7.

We have then shown that the irrational roots of a quadratic equation with real coefficients can, with certain restrictions, be expressed as functions of the same periodic continued fractions. A quadratic equation with a positive and negative root always has the continued fraction expressions for its roots as defined in theorem 4.2.1. The coefficients of an equation with two positive or two negative roots must satisfy certain conditions or else the theorem covering these cases does not apply.

Corollary 4.2.1

If a quadratic equation with the conditions of theorem 4.2.1 has one root of the form

$$x = -c\left(\cfrac{1}{b-\cfrac{ac}{b-\cfrac{ac}{b-\ddots}}}\right),$$

the other is of the form

$$x = -\frac{1}{a}\left(b-\cfrac{ac}{b-\cfrac{ac}{b-\ddots}}\right).$$

This is easily proven by (4.2.5) or (4.2.6). The converse is also true.

Corollary 4.2.2

$$\alpha = \frac{b \pm \sqrt{b^2 - 4ac}}{2}$$

Since x_1 and x_2 are the two roots of the general quadratic equation, we have

$$x_1 + x_2 = -\frac{b}{a}$$

or

$$-\frac{c}{\alpha} - \frac{\alpha}{a} = -\frac{b}{a}$$

$$-ac - \alpha^2 = -b\alpha$$

$$\alpha^2 - b\alpha + ac = 0$$

$$\alpha = \frac{b \pm \sqrt{b^2 - 4ac}}{2}.$$

If we take

$$\alpha = \frac{b + \sqrt{b^2 - 4ac}}{2}$$

then

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad , \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad ;$$

and if we take

$$\alpha = \frac{b - \sqrt{b^2 - 4ac}}{2}$$

then

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad , \quad x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad .$$

Now, then, is the above theory still applicable if the equation has imaginary roots? Using the development as in theorem 4.2.1, we find that the continued fractions generated will be second class continued fractions. By theorem 3.7.., they will converge if $b \geq ac+1$. But we are given $b^2 < 4ac$ so that

$$b^2 \geq (ac+1)^2 \geq 4ac > b^2$$

for all $ac > 0$ and we see that given $b^2 < 4ac$, we cannot find rational numbers such that

$$b \geq ac+1$$

and hence the theory is not applicable.

Theorem 4.2.4 Theorem of Galois[1]

If one of the roots of an equation of any degree is a pure periodic continued fraction (all the elements are positive) this equation will necessaily have another root obtained by dividing minus one by a pure periodic continued fraction whose elements are the same as the original fraction but written in reverse order.

This theorem is, of course, directly associated with quadratic equations. If we take the original root to be

$$x = a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{a+\cfrac{1}{b+\ \cdots}}}}} \qquad ,$$

we have

$$x = a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{x}}}}$$

and

$$a - x = -\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{x}}}} \qquad ,$$

1. Galois, Ouvres Mathématiques D'Évariste Galois, pages 1-12

$$\frac{1}{a-x} = -b-\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{x}}} \quad ,$$

$$b+\frac{1}{a-x} = -\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{x}}} \quad ,$$

$$\cfrac{1}{b+\cfrac{1}{a-x}} = -c-\cfrac{1}{d+\cfrac{1}{x}} \quad ,$$

$$c+\cfrac{1}{\cfrac{1}{a-x}} = -\cfrac{1}{d+\cfrac{1}{x}} \quad ,$$

$$\cfrac{1}{c+\cfrac{1}{b+\cfrac{1}{a-x}}} = -d-\frac{1}{x} \quad ,$$

$$d+\cfrac{1}{c+\cfrac{1}{b+\cfrac{1}{a-x}}} = -\frac{1}{x} \quad ,$$

$$\cfrac{1}{d+\cfrac{1}{c+\cfrac{1}{b+\cfrac{1}{a-x}}}} = -x \quad ,$$

therefore,

$$x = -\cfrac{1}{d+\cfrac{1}{c+\cfrac{1}{b+\cfrac{1}{a+\cfrac{1}{d+\cfrac{1}{c+\ddots}}}}}}$$

is another root of the equation.

Example 1

Solve the equation

$$x^2 + 2x - 1 = 0.$$

Since the equation has one variation of sign and

$$b^2 - 4ac > 0\ ,$$

we can apply theorem 4.2.1. In approximating roots by this method, it is best to obtain the approximant of α and substitute that value in x_1 and x_2.

$$x_1 = \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}$$

and

$$x_2 = -\left(2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}\right).$$

$$\alpha_5 = \frac{169}{70} = 2.41428$$

$$x_2 = -2.41428$$

$$x_1 = \frac{70}{169} = 0.41428.$$

The errors for the approximations are

$$|X_1 - x_1| < \frac{1}{70^2}$$

$$|X_2 - x_2| < \frac{1}{70^2} \quad .$$

By the quadratic formula

$$x_1 = -1 + \sqrt{2}$$

$$x_2 = -1 - \sqrt{2} \quad .$$

Since

$$x_1 = \cfrac{1}{2+\cfrac{1}{2+\;.\;.\;.}} \quad ,$$

then

$$\sqrt{2} = 1 + \cfrac{1}{2+\cfrac{1}{2+\;.\;.\;.}} \quad .$$

Example 2 .

To develop the continued fraction expression for the golden section.

The name golden section is applied to the division of a line segment in the following manner:

> "To cut a given line in extreme and mean ratio ." (as found in Euclid's Elements).

This problem is also contained in the construction of the regular inscribed decagon and we shall develop the continued fraction from it.[1]

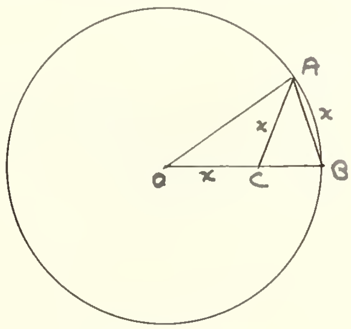

Figure 2

Using a circle of unit radius, we wish to determine the side x of the decagon. We have

$$\angle O = 36^\circ \text{ , } \angle OAB = 72^\circ \text{ , } \angle ABO = 72^\circ \text{ .}$$

Construct AC bisecting $\angle OAB$. Therefore,

$$\angle ACB = \angle ABC \text{ ,}$$

$$AB = AC = OC = x,$$

$$CB = 1-x \text{ ,}$$

$$\triangle OAB \cong \triangle ACB \text{ ,}$$

1. Courant and Robbins, What is Mathematics?, page 123.

and we have

$$\frac{1}{x} = \frac{x}{1-x}$$

$$x^2 + x - 1 = 0 \, .$$

The ratio

$$\frac{1-x}{x}$$

is the required extreme and mean ratio.

The positive root of the equation

$$x^2 + x - 1 = 0$$

is

$$x_1 = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+ \cdots}}}$$

which has for its successive approximants

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \ldots \, .$$

By the quadratic formula

$$\frac{\sqrt{5}-1}{2} = \cfrac{1}{1+\cfrac{1}{1+ \cdots}} \, ,$$

the value of whichis usually taken as 0.618. This is also the value of

$$\frac{1-x}{x} \quad .$$

In Dynamic Symmetry, by J. Hambidge, it is claimed that the use of the golden section by the Greeks enabled them to achieve the beautiful proportions found in their temples, sculpture and vases. Psychologists have experimentally proven that the rectangle of most pleasing proportions was one in which the adjacent sides are in the ratio of parts of a line segment divided in golden section.

It might be pointed out that

$$\sum_{n=0}^{\infty} A_n \, ,$$

where A_n is the nth numerator of x_1 , is the Fibonacci series

$$0 + 1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + \dots \quad .$$

4.3

Solve the exponential equation

$$2^x = 6 \, .$$[1]

Since x must be greater than 2 and less than 3, we have

1. Davis and Peck, Mathematical Dictionary, pages 240-41.

$$x = 2 + \frac{1}{a} \qquad a > 1 \;,$$

$$(2)^{2+1/a} = 6 \;,$$

$$2^2 \cdot 2^{1/a} = 6 \;,$$

$$2^{1/a} = \frac{3}{2} \;,$$

$$\left(\frac{3}{2}\right)^a = 2 \;.$$

Here a must be greater than 1 and less than 2.

$$a = 1 + \frac{1}{b} \qquad b > 1 \;,$$

$$\left(\frac{3}{2}\right)^{1+1/b} = 2 \;,$$

$$\left(\frac{4}{3}\right)^b = \frac{3}{2} \;.$$

$$b = 1 + \frac{1}{c} \qquad c > 1 \;,$$

$$\left(\frac{9}{8}\right)^c = \frac{4}{3} \;.$$

$$c = 2 + \frac{1}{d} \qquad d > 1 \;,$$

$$\left(\frac{256}{243}\right)^d = \frac{9}{8} \;.$$

$$d = 2 + \frac{1}{e} \qquad e > 1 \;.$$

We then have

$$x = 2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{e+\ddots}}}}}$$

and

$$\frac{A_4}{B_4} = \frac{31}{12} = 2.5833\ldots \quad .$$

By logarithms

$$x = 2.5849\ldots \quad .$$

4.4

Many applications for continued fractions have been found in the solution of Diophantine problems.[1] We shall only introduce the problem of finding the integral solutions of the indeterminate equation

(4.4.1) $$ax \pm by = c$$

where <u>a</u> and <u>b</u> are integers prime to each other and <u>c</u> is an integer.

If we convert a/b into a simple continued fraction, its last approximant will be a/b. Letting the penultimate approximant be p/q, we have by (1.6.2)

(4.4.2) $$aq - bp = \pm 1 \quad .$$

1. Chrystal, <u>Algebra</u>, Volume 2, pages 445-448.

Therefore

$$a(\pm cq) - b(\pm cp) = c \tag{4.4.3}$$

and hence,

$$x' = \pm cq \;, \quad y' = \pm cp$$

is a particular integral solution. Subtracting (4.4.3) from (4.4.1), we obtain

$$a[x-(\pm cq)] - b[y-(\pm cp)] = 0 \;,$$

$$\frac{x-(\pm cq)}{y-(\pm cp)} = \frac{b}{a} \quad .$$

Since a is prime to b, it follows that

$$x-(\pm cq) = bt$$

and

$$y-(\pm cp) = at$$

where t is any integer. Therefore, every integral solution of (4.4.1) is included in

$$x = \pm cq + bt$$

$$y = \pm cp + at$$

where the sign depends on the one taken in (4.4.2).

Example:

To find all the integral solutions of

$$8x + 13y = 159.$$

$$\frac{8}{13} = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}}}$$

and the penultimate approximant is 3/5. Then

$$8 \cdot 5 - 13 \cdot 3 = 1$$

$$8(795) + 13(-477) = 159$$

and

$$x' = 795 \ , \quad y' = -477$$

are particular solution and the general solution is

$$x = 795 - 13t$$
$$y = -477 + 8t \quad .$$

CHAPTER V

Analytic Continued Fractions

By the term analytic continued fraction, is meant a continued fraction in which the elements are analytic functions of a single variable or of several variables. We will first establish a continued fraction expansion for any series and from this a corresponding expression for the power series.

5.1 Infinite Series.

Continued fractions and series were first related by the method employed in corollaries 2.3.3 and 3.2.4 where it was established that

$$\frac{A_q}{B_q} = b_0 + \frac{1}{B_0 B_1} - \frac{1}{B_1 B_2} + \cdots + \frac{(-1)^{q-1}}{B_{q-1} B_q}$$

and

$$\frac{A_q}{B_q} = b_0 + \frac{a_1}{B_0 B_1} - \frac{a_1 a_2}{B_1 B_2} + \cdots + \frac{(-1)^{q-1} a_1 \cdots a_q}{B_{q-1} B_q} \quad .$$

We will now introduce a method which will enable us to transform an infinite series into an equivalent continued fraction. A continued fraction is said to be equivalent to a series when the nth approximant of the former is equal to the sum of <u>n</u> terms of the latter for all values of <u>n</u> .

Theorem 5.1[1]

The infinite series

$$u_0 + u_1 + \dots + u_n + \dots$$

can be converted into an equivalent continued fraction of the form

(5.1.1)
$$b_0 + \cfrac{a_1}{b_1 - \cfrac{a_2}{b_2 - \cfrac{a_3}{b_3 - \dots}}} \quad .$$

For the fraction (5.1.1), we have by corollary (3.2.3)

(5.1.2) $$\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} = \frac{a_1 a_2 \cdots a_n}{B_{n-1} B_n} \quad ;$$

$$B_0 = 1 \ , \quad B_1 = b_1 B_0 \ , \quad B_2 = b_2 B_1 - a_2 B_0 \ ,$$

(5.1.3) $$B_n = b_n B_{n-1} - a_n B_{n-2} \ ;$$

$$\frac{A_0}{B_0} = \frac{b_0}{B_0} = b_0 \quad .$$

Since

$$\frac{A_n}{B_n} = u_0 + u_1 + \dots + u_n \quad ,$$

1. Chrystal, Algebra, Volume II, pages 486-489.

we obtain from (5.1.2) and (5.1.3)

$$\frac{A_0}{B_0} = u_0 = \frac{b_0}{B_0} \ ,$$

$$\frac{A_1}{B_1} - \frac{A_0}{B_0} = u_1 = \frac{a_1}{B_0 B_1}$$

(5.1.4) $$\frac{A_2}{B_2} - \frac{A_1}{B_1} = u_2 = \frac{a_1 a_2}{B_1 B_2}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} = u_n = \frac{a_1 a_2 \cdots a_n}{B_{n-1} B_n} \ .$$

From the relations (5.1.4), we have

$$b_0 = u_0, \ a_1 = B_1 u_1 \ , \ a_2 = \frac{B_2 u_2}{B_0 u_1}, \ a_3 = \frac{B_3 u_3}{B_1 u_2}$$

(5.1.5) $$a_n = \frac{B_n u_n}{B_{n-2} u_{n-1}} \ .$$

By combining (5.1.3) and (5.1.5), we obtain

$$b_0 = u_0, \ b_1 = \frac{B_1}{B_0}, \ b_2 = \frac{B_2(u_1+u_2)}{B_1 u_1} \ ,$$

(5.1.6) $$b_3 = \frac{B_3(u_2+u_3)}{B_2 u_2} \ , \ldots, \ b_n = \frac{B_n(u_{n-1}+u_n)}{B_{n-1} u_{n-1}} \ .$$

We then have

$$S_n = u_0 + u_1 + \dots + u_n$$

$$S_n = u_0 + \cfrac{B_1 u_1}{B_1 - \cfrac{B_2 u_2/u_1}{B_2(u_1+u_2)/B_1u_1 - \cfrac{B_3u_3/B_1u_2}{B_3(u_2+u_3)/B_2u_2 - \ddots \cfrac{B_n u_n/B_{n-2}u_{n-1}}{B_n(u_{n-1}+u_n)/B_{n-1}u_{n-1}}}}} \quad . \tag{5.1.7}$$

This fraction can be transformed into

$$S_n = u_0 + \cfrac{u_1}{1 - \cfrac{u_2}{u_1+u_2 - \cfrac{u_1u_3}{u_2+u_3 - \ddots \cfrac{u_{n-2}u_n}{u_{n-1}+u_n}}}} \quad . \tag{5.1.8}$$

By taking the successive approximants of (5.1.7) or (5.1.8), we see that the value of these expressions are equal to the partial sums of the series for the same value of n. Then, $\lim_{n\to\infty} S_n$ for the series and for (5.1.8) must be the same. Therefore, if the series converges, the fraction must converge to the same limit, and conversely.

By application of theorem 5.1.1, we can obtain the following relationships:

$$k_0 + k_1 x + k_2 x^2 + \cdots + k_n x^n + \cdots \qquad (k_i \neq 0)$$

$$= k_0 + \cfrac{k_1 x}{1 - \cfrac{k_2 x}{k_1 + k_2 x - \cfrac{k_1 k_3 x}{k_2 + k_3 x - \ddots - \cfrac{k_{n-2} k_n x}{k_{n-1} + k_n x - \ddots}}}} ,$$

$$k_0 + \frac{x}{k_1} + \frac{x^2}{k_2} + \cdots + \frac{x^n}{k_n} + \cdots \qquad (k_i \neq 0)$$

$$= k_0 + \cfrac{x}{k_1 - \cfrac{k_1^2 x}{k_1 x + k_2 - \cfrac{k_2^2 x}{k_2 x + k_3 - \ddots - \cfrac{k_{n-1}^2 x}{k_{n-1} x + k_n - \ddots}}}} ,$$

$$h_0 + \frac{h_1}{k_1} x + \frac{h_1 h_2}{k_1 k_2} x^2 + \cdots + \frac{h_1 h_2 \cdots h_n}{k_1 k_2 \cdots k_n} x^n + \cdots \qquad (h_i, k_i \neq 0)$$

$$= h_0 + \cfrac{h_1 x}{k_1 - \cfrac{k_1 h_2 x}{k_2 + h_2 x - \cfrac{k_2 h_3 x}{k_3 + h_3 x - \ddots - \cfrac{k_{n-1} h_n x}{k_n + h_n x - \ddots}}}} .$$

Example

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

$$= \cfrac{x}{1+\cfrac{1^2x^2}{3-x^2+\cfrac{3^2x^2}{5-3x^2+\cfrac{5^2x^2}{7-5x^2+\cdots}}}} \quad ;$$

and for $x = 1$

$$\frac{\pi}{4} = \cfrac{1}{1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cdots}}}}$$

which is Lord Brouncker's formula.

5.2 The Continued Fraction of Gauss.[1]

In 1812, Gauss published a paper[2] in which he employed the hypergeometric series

$$(5.2.1) \quad F(a,b,c;z) = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{2!c(c+1)}z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)}z^3 + \cdots,$$

where a and b are any complex constants and c is a complex constant different from $0,-1,-2,-3,\ldots,$ in

1. Wall, Continued Fractions, Chapter XVIII.
2. Gauss, Werkes, Volume II, pages 125-160.

obtaining continued fraction expressions for such functions as log(1+z), arctan z, e^z, etc.. If a or b is zero or a negative integer, (5.2.1) reduces to a polynomial, otherwise, it is an infinite series with radius of convergence equal to unity. By varying the values of the parameters, (5.2.1) reduces to elementary functions in a number of cases. For example

$$F(1,1,2;-z) = 1 - \frac{1}{2}z + \frac{1}{3}z^2 - \frac{1}{4}z^3 + \dots$$

which is the Maclaurin expansion for

$$\frac{1}{z}\log(1+z)\,. \tag{5.2.2}$$

Similarly, we have

$$F(-K,1,1;-z) = (1+z)^k\,, \tag{5.2.3}$$

$$zF(1/2,1/2,3/2;z^2) = \arcsin z\,, \tag{5.2.4}$$

$$zF(1/2,1,3/2;-z^2) = \arctan z\,, \tag{5.2.5}$$

$$2zF(1/2,1,3/2;z^2) = \log\frac{1+z}{1-z}\,. \tag{5.2.6}$$

We can also establish the following:

Replacing z by z/a and letting a tend to ∞, the series (5.2.1) becomes

$$G(b,c,;z) = 1 + \frac{b}{c}z + \frac{b(b+1)}{c(c+1)}\frac{z^2}{2!} + \frac{b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \dots, \tag{5.2.7}$$

and in (5.2.7), by replacing z by z/b and letting b tend

to infinity we have

$$(5.2.8)\quad H(c;z) = 1+ \frac{1}{c}\, z+ \frac{1}{c(c+1)}\, \frac{z^2}{2!} + \frac{1}{c(c+1)(c+2)}\, \frac{z^3}{3!}+\dots,$$

and finally by replacing z by cz in (5.2.1), we have

$$(5.2.9)\qquad K(a,b;z) = 1+ abz+a(a+1)b(b+1)\frac{z^2}{2!}$$
$$+ a(a+1)(a+2)b(b+1)(b+2)\frac{z^3}{3!} +\dots \quad .$$

(5.2.9) has a zero radius of convergence except when a or b is a negative integer or zero.

For the series (5.2.1), we have the relationship

(5.2.10)

$$F(a,b,c;z) = F(a,b+1,c+1;z) - \frac{a(c-b)}{c(c+1)}\, zF(a+1,b+1,c+2;z).$$

$$1 + \frac{ab}{c}\, z+\dots = 1+ \frac{a(b+1)}{(c+1)}\, z+\dots - \frac{a(c-b)}{c(c+1)}\, z-\dots$$

$$= 1+ \frac{abc+ac-ac+ab}{c(c+1)}\, z+\dots$$

$$= 1+ \frac{ab}{c}\, z+\dots$$

and in like manner we can show that the coefficients of corresponding powers of z are equal and the identity holds. We may write (5.2.10) as

$$\frac{F(a,b,c;z)}{F(a,b+1,c+1;z)} = 1- \frac{a(c-b)}{c(c+1)}\, z\, \frac{F(a+1,b+1,c+2;z)}{F(a,b+1,c+1;z)}$$

or

(5.2.11)

$$\frac{F(a,b+1,c+1;z)}{F(a,b,c;z)} = \cfrac{1}{1-\frac{a(c-b)}{c(c+1)}\,z\,\frac{F(a+1,b+1,c+2;z)}{F(a,b+1,c+1;z)}} \quad .$$

In (5.2.11) interchange a and b so that

$$\frac{F(b,a+1,c+1;z)}{F(b,a,c;z)} = \cfrac{1}{1-\frac{b(c-a)}{c(c+1)}\,z\,\frac{F(b+1,a+1,c+2;z)}{F(b,a+1,c+1;z)}} \quad .$$

Let b,=b+1, c=c+1, and then interchange a and b again to obtain

(5.2.12)

$$\frac{F(a+1,b+1,c+2;z)}{F(a,b+1,c+1;z)} = \cfrac{1}{1-\frac{(b+1)(c-a+1)}{(c+1)(c+2)}\,z\,\frac{F(a+1,b+2,c+3;z)}{F(a+1,b+1,c+2;z)}} \quad .$$

The quotient in the left-hand member of (5.2.12) is the same as the quotient of hypergeometric series appearing in the denominator of the right-hand member of (5.2.11). Also, if a,b,c are replaced by a+1,b+1,c+2, respectively, in (5.2.11), the quotient in the left-hand member becomes equal to the quotient of hypergeometric series appearing in the denominator of the right-hand member of (5.2.12). On applying first

one identity and then the other, we obtain by successive substitution the continued fraction of Gauss:

$$\frac{F(a,b+1,c+1;z)}{F(a,b,c;z)} = \cfrac{1}{1-\cfrac{\frac{a(c-b)}{c(c+1)}z}{1-\cfrac{\frac{(b+1)(c-a+1)}{(c+1)(c+2)}z}{1-\cfrac{\frac{(a+1)(c-b+1)}{(c+2)(c+3)}z}{1-\cfrac{\frac{(b+2)(c-a+2)}{(c+3)(c+4)}z}{1-\cfrac{\frac{(a+2)(c-b+2)}{(c+4)(c+5)}z}{1-\ddots}}}}}}$$

Letting $a_1, a_2, a_3, \ldots$ denote the coefficients of $-z$ in the partial numerators, we have

$$a_{2q+1} = \frac{(a+q)(c-b+q)}{(c+2q)(c+2q+1)}, \qquad q=0,1,2,\ldots$$

$$a_{2q+2} = \frac{(b+q+1)(c-a+q+1)}{(c+2q+1)(c+2q+2)} .$$

If we put

$$P_{2n}(z) = \frac{F(a+n,b+n+1,c+2n+1;z)}{F(a+n,b+n,c+2n;z)}, \qquad n=0,1,2,\ldots$$

$$P_{2n+1}(z) = \frac{F(a+n+1,b+n+1,c+2n+2;z)}{F(a+n,b+n+1,c+2n+1;z)},$$

we then have

$$P_{n-1}(z) = \frac{1}{1-a_n z P_n(z)}, \qquad n=1,2,3,\ldots .$$

Hence, for every $\underline{n}$,

$$(5.2.14) \qquad \frac{F(a,b+1,c+1;z)}{F(a,b,c;z)} = \cfrac{1}{1-\cfrac{a_1 z}{1-\cfrac{a_2 z}{1- \ddots \cfrac{a_{n-1} z}{1-a_n z P_n(z)}}}} .$$

If $a_1, a_2, \ldots, a_{n-1}$ are different from zero, while $a_n = 0$, then the continued fraction of Gauss terminates, and the quotient

$$\frac{F(a,b+1,c+1;z)}{F(a,b,c;z)}$$

is a rational function of $\underline{z}$, which is equal to the terminating continued fraction. If, on the other hand, $a_q \neq 0$, $q = 1,2,3,\ldots$, we may write (5.2.14) in the form

$$\frac{F(a,b+1,c+1;z)}{F(a,b,c;z)} = \frac{A_n(z)-a_n z P_n(z) A_{n-1}(z)}{B_n(z)-a_n z P_n(z) B_{n-1}(z)} ,$$

where $A_q(z)$ and $B_q(z)$ are the qth numerator and denominator of the continued fraction of Gauss. Then

$$\frac{F(a,b+1,c+1;z)}{F(a,b,c;z)} - \frac{A_{n-1}(z)}{B_{n-1}(z)} = \frac{A_n(z)B_{n-1}(z)-A_{n-1}(z)B_n(z)}{B_{n-1}(z)[B_n(z)-a_n z P_n(z) B_{n-1}(z)]}$$

$$= \frac{a_1 a_2 \cdots a_{n-1} z^{n-1}}{B_{n-1}(z)[B_n(z)-a_n z P_n(z) B_{n-1}(z)]} .$$

This shows that the power series in ascending powers of $\underline{z}$ in $A_{n-1}(z)/B_{n-1}(z)$ agrees with the power series

$$\frac{F(a,b+1,c;z)}{F(a,b,c;z)}$$

term by term for the first $\underline{n}$ terms. Therefore, the power series for the hypergeometric quotient is the power series expansion of the continued fraction of Gauss.

For the proof of the uniform convergence of (5.2.14), the reader is referred to Wall, Continued Fractions, pages 339.

If we let b = 0 and replace $\underline{c}$ by c - 1, then the continued fraction (5.2.13) becomes

$$F(a,1,c;z) = \cfrac{1}{1-\cfrac{\frac{a}{c}z}{1-\cfrac{\frac{c-a}{c(c+1)}z}{1-\cfrac{\frac{c(a+1)}{(c+1)(c+2)}z}{1-\cfrac{\frac{2(c-a+1)}{(c+2)(c+3)}z}{1-\cfrac{\frac{(c+1)(a+2)}{(c+3)(c+4)}z}{1-\ddots}}}}}} \qquad (5.2.15)$$

5.3 Representation of Functions by Analytic Continued Fractions.

By (5.2.2) and (5.2.15) with a=1, c=2, z=-z, we have

$$\log(1+z) = \cfrac{z}{1-\cfrac{1/2(-z)}{1-\cfrac{1/2\cdot 3(-z)}{1-\cfrac{2\cdot 2/3\cdot 4(-z)}{1-\cfrac{2\cdot 2/4\cdot 5(-z)}{1- \ddots}}}}}$$

$$= \cfrac{z}{1+\cfrac{1^2 z}{2+\cfrac{1^2 z}{3+\cfrac{2^2 z}{4+\cfrac{2^2 z}{5+\cfrac{3^2 z}{6+\ddots}}}}}} \tag{5.3.1}$$

for $z > -1$.

By (5.2.3) and (5.2.15) with a = -k, c=1, z=-z, we have

$$(1+z)^k = \cfrac{1}{1-\cfrac{-k(-z)}{1-\cfrac{(1+k)/2(-z)}{1-\cfrac{(-k+1)/2\cdot 3(-z)}{1-\cfrac{2(2+k)/3\cdot 4(-z)}{1- \ddots}}}}}$$

$$= \cfrac{1}{1-\cfrac{kz}{1+\cfrac{\frac{1\cdot(1+k)}{1\cdot 2}z}{1+\cfrac{\frac{1\cdot(1-k)}{2\cdot 3}z}{1+\cfrac{\frac{2(2+k)}{3\cdot 4}z}{1+\cfrac{\frac{2(2-k)}{4\cdot 5}z}{1+\cfrac{\frac{3(3+k)}{5\cdot 6}z}{1+\cdots}}}}}}} \tag{5.3.2}$$

for $z > -1$, and by (5.2.5) and (5.2.15) with $a=1/2$, $c=3/2$, $z=-z^2$ we have

$$\arctan z = \cfrac{z}{1+\cfrac{1^2 z^2}{3+\cfrac{2^2 z^2}{5+\cfrac{3^2 z^2}{7+\cfrac{4^2 z^2}{9+\cdots}}}}} \tag{5.3.3}$$

for all values of z.

Also by (5.2.6) and (5.2.15), we have

$$\log \frac{1+z}{1-z} = \cfrac{2z}{1-\cfrac{1^2 z^2}{3-\cfrac{2^2 z^2}{5-\cfrac{3^2 z^2}{7-\cdots}}}} \tag{5.3.4}$$

for $|z| > 1$, and by replacing z by $1/z$ we obtain

$$\log\frac{z+1}{z-1} = \cfrac{2}{z-\cfrac{1/2}{\frac{3}{2}z-\cfrac{2/3}{\frac{5}{3}z-\cfrac{3/4}{\frac{7}{4}z-\cdots}}}} \tag{5.3.5}$$

for $|z|>1$.

We have

$$\log\frac{z+1}{z-1} = \int_{-1}^{+1}\frac{dx}{z-x} = (5.3.5)\ .$$

In (5.2.13), let $z=z/a$ and $\underline{a}$ tend to ∞.

We then obtain the continued fraction

$$\frac{G(b+1,c+1;z)}{G(b,c;z)} = \cfrac{1}{1-\cfrac{\frac{(c-b)}{c(c+1)}z}{1+\cfrac{\frac{(b+1)}{(c+1)(c+2)}z}{1-\cfrac{\frac{(c-b+1)}{(c+2)(c+3)}z}{1+\cfrac{\frac{(b+2)}{(c+3)(c+4)}z}{1-\cdots}}}}}\ . \tag{5.3.8}$$

Setting $b=0$, (5.3.8) becomes

$$G(1,c;z) = \cfrac{1}{1-\cfrac{z}{c+\cfrac{1\cdot z}{c+1-\cfrac{cz}{c+2+\cfrac{2z}{c+3-\cfrac{(c+1)z}{c+4+\cfrac{3z}{c+5-\cdots}}}}}}}\ . \tag{5.3.9}$$

With $c=1$ we have, with the aid of an equivalence transformation,

$$e = 1+z+\frac{1}{2!}z^2+\frac{1}{3!}z^3+\ldots$$

$$= \cfrac{1}{1-\cfrac{z}{1+\cfrac{z}{2-\cfrac{z}{3+\cfrac{z}{2-\cfrac{z}{5+\cfrac{z}{2-\ddots}}}}}}}$$

for $|z| \geq 0$.

In (5.3.8) let $z=z/b$ and $\underline{b}$ tend to ∞. Then (5.5.23) becomes

$$\frac{H(c+1;z)}{H(c;z)} = \cfrac{1}{1+\cfrac{\frac{z}{c(c+1)}}{1+\cfrac{\frac{z}{(c+1)(c+2)}}{1+\cfrac{\frac{z}{(c+2)(c+3)}}{1+\cfrac{\frac{z}{(c+3)(c+4)}}{1+\ddots}}}}} \tag{5.3.10}$$

for $|z| \geq 0$.

For $c=3/2$ and $z=z/4$ we have

$$\frac{e^z-e^{-z}}{e^z+e^{-z}} = \frac{zH(3/2;z/4)}{H(1/2;z/4)} = \cfrac{z}{1+\cfrac{z^2}{3+\cfrac{z^2}{5+\cfrac{z^2}{7+\cdots}}}} \tag{5.3.11}$$

for $|z| \geq 0$.

5.4 Divergent Series and Continued Fractions.[1]

On replacing $\underline{z}$ by $-cz$ in (5.2.13) and then letting $\underline{c}$ tend to ∞, we obtain for the quotient of two of the divergent series (5.2.9) the expansion

$$\frac{K(a,b;-z)}{K(a,b-1;-z)} = \cfrac{1}{1+\cfrac{az}{1+\cfrac{bz}{1+\cfrac{(a+1)z}{1+\cfrac{(b+1)z}{1+\cfrac{(a+2)z}{1+\cfrac{(b+2)z}{1+\cdots}}}}}}} . \tag{5.4.1}$$

(Here, the equality sign is purely formal). If $b=1$, we have

$$1-az+a(a+1)z^2-a(a+1)(a+2)z^3+\ldots$$

1. Wall, Continued Fractions, pages, 349-355
Van Vleck, Boston Colloquium, pages 93-94.

$$= k(a,1;-z) = \cfrac{1}{1+\cfrac{az}{1+\cfrac{1\cdot z}{1+\cfrac{(a+1)z}{1+\cfrac{2\cdot z}{1+\cfrac{(a+2)z}{1+\ddots}}}}}} \quad . \tag{5.4.2}$$

The continued fractions in (5.4.1) and (5.4.2) can be shown to be convergent for $z > -1$ even though the power series involved have a zero radius of convergence. For the proof of the convergence, the reader is referred to Wall, Continued Fractions, pages 350-351. From the above relationships we can, in a sense, sum a divergent power series by means of an "equivalent" convergent continued fraction. We may add this method of summing a divergent infinite series to the methods of Cesaro, Abel and Hölder.

The series (5.4.2) with a=1 becomes

$$1 - z + 2!z^2 - 3!z^3 + \ldots \tag{5.4.3}$$

which is a special case of the series

$$1+mz+m(m+n)z^2+m(m+n)(m+2n)z^3 + \ldots \quad .$$

Both of the above series have a radius of convergence equal to zero. Since

$$m! = \Gamma(m+1) = \int_0^\infty e^{-x} x^m dx \quad ,$$

the series (5.4.3) may be written

$$\int_0^\infty e^{-x} dx - z\int_0^\infty e^{-x} x dx + z^2\int_0^\infty e^{-x} x^2 dx - \ldots \quad .$$

If the sum of the integrals is replaced by the integral of the sum, we obtain

$$\int_0^\infty e^{-x}(1-zx + z^2x^2 - \ldots)dx$$

or a function

$$f(z) = \int_0^\infty e^{-x} F(zx)dx$$

in which

$$F(zx) = \frac{1}{1+zx} \quad .$$

Then, by (5.4.2), with a=1, we have

$$1-z+2!z^2 - 3!z^3 + \ldots = \int_0^\infty e^{-x} \frac{dx}{1+zx}$$

$$= \cfrac{1}{1+\cfrac{z}{1+\cfrac{z}{1+\cfrac{2z}{1+\cfrac{2z}{1+\cfrac{3z}{1+\ddots}}}}}} \quad .$$

This is a special case of

$$\frac{1}{\Gamma(a)}\int_0^\infty \frac{e^{-x}\,x^{a-1}\,dx}{1+zx} = \cfrac{1}{1+\cfrac{az}{1+\cfrac{1\cdot z}{1+\cfrac{(a+1)z}{1+\cfrac{2z}{1+\cfrac{(a+2)z}{1+\ddots}}}}}}$$

which is valid for $a > 0$ and $z > 0$.

5.5 Orthogonal Polynomials and Continued Fractions.[1]

Historically, the orthogonal polynomials originated in the study of continued fractions. It was discovered that the denominators (of the approximants) of the continued fraction

$$\cfrac{a_1}{b_1+x-\cfrac{a_2}{b_2+x-\cdots\cfrac{a_q}{b_q+x-\ddots}}}$$

form a sequence of polynomials $B_q(x)$ such that

$$(5.5.1)\quad \begin{aligned} &\int_a^b B_m(x)B_n(x)\,dx=0 \quad \text{for } m\neq n, \quad m,n \leq q; \\ &\int_a^b B_m(x)B_n(x)\,dx\neq 0 \quad \text{for } m=n, \quad m=n \leq q; \end{aligned}$$

1. Szegö, <u>Orthogonal Polynomials</u>, pages 41,53-56.

and

$$B_0(x) = 1, \quad B_1(x) = b_1 + x,$$

$$B_n(x) = (b_n+x)B_{n-1}(x) - a_n B_{n-2}(x),$$
$$n=2,3,\ldots,q.$$

From the theory of orthogonal polynomials we have

Theorem 5.5.1 (Christoffel-Darboux formula).

The following relations hold for any three consecutive orthogonal polynomials:

$$(5.5.2) \quad P_n(x) = (C_n x+D_n)P_{n-1}(x) - E_n P_{n-2}(x), \quad P_{-1}(x) = 0, \qquad n=1,2,3,\ldots .$$

Here C_n, D_n, and E_n are constants, $C_n > 0$ and $E_n > 0$.

For the continued fraction

$$b_0 + \cfrac{a_1}{b_1 + \cdots \cfrac{a_n}{b_n + \cdots}} ,$$

we have the recurrence formulas

$$(5.5.3) \quad \begin{aligned} A_n &= b_n A_{n-1} + a_n A_{n-2} \\ B_n &= b_n B_{n-1} + a_n B_{n-2} \end{aligned} .$$

The formulas (5.5.2) and (5.5.3) suggest the considera-tion of the continued fraction

(5.5.4) $$\cfrac{1}{C_1x+D_1-\cfrac{E_2}{C_2x+D_2-\cdot\cdot\cfrac{E_n}{C_nx+D_n-\cdots}}}$$

with

$$b_0 = 0, \quad b_n = D_n + C_n x; \quad a_1 = 1, a_n = -E_n \ .$$

For (5.5.4) we then have

$$B_n(x) = (C_n x + D_n)B_{n-1} - E_n B_{n-2}(x)$$

with

$$B_{-1}(x) = 0, \quad B_0(x) = 1, \quad B_1(x) = C_1 x + D_1 \ .$$

However, since this relationship is exactly the Christoffel-Darboux formula, we conclude that the denominators, B_n , of (5.5.4) are orthogonal polynomials.

The continued fraction

$$\cfrac{2}{z-\cfrac{1/2}{\frac{3}{2}z-\cfrac{2/3}{\frac{5}{3}z-\cfrac{3/4}{\frac{7}{4}z-\cdots}}}}$$

which is the continued fraction expansion for $\log(z+1)/(z-1)$, see (5.3.5), has for its denominators the polynomials

$$B_0 = 1,$$
$$B_1 = z,$$
$$B_2 = \frac{3}{2} zB_1 - \frac{1}{2} B_0$$
$$= \frac{1}{2}(3z^2-1),$$
$$B_3 = \frac{1}{2}(5z^3-z),$$
$$B_4 = \frac{1}{8}(35z^4-30z^2+3), \; \ldots ,$$

which are the Legendre polynomials. These polynomials have the property that

$$\int_{-1}^{+1} B_m(x)\, B_n(x)\, dx = 0$$

for $m \neq n$. We then have in continued fractions a fairly simple means of calculating these polynomials.

5.6 Solution of a Second Order Differential Equation by Continued Fractions.[1]

Let us consider the equation

(5.6.1) $$y = Q_0 y' + P_1 y'' ,$$

where Q_0 and P_1 are functions of $\underline{x}$. Differentiating (5.6.1), we obtain

1. Ince, Ordinary Differential Equations, pages 178-180.

$$y' = Q_0 y'' + Q_0' y' + P_1 y''' + P_1' y''$$

$$y' = Q_1 y'' + P_2 y''',$$

where

$$Q_1 = \frac{Q_0 + P_1'}{1-Q_0'}, \quad P_2 = \frac{P_1}{1-Q_0'} .$$

By repeating this process indefinitely, we obtain the relations

(5.6.2) $$y^{(n)} = Q_n y^{(n+1)} + P_{n+1} y^{(n+2)}$$

where $n=1,2,3,\ldots,$ and

(5.6.3) $$Q_n = \frac{Q_{n-1} + P_n'}{1-Q_{n-1}'}, \quad P_{n+1} = \frac{P_n}{1-Q_{n-1}'} .$$

From (5.6.1) and (5.6.2), we have

$$\frac{y}{y'} = Q_0 + P_1 \frac{y''}{y'} = Q_0 + \cfrac{P_1}{\cfrac{y'}{y''}}$$

$$= Q_0 + \cfrac{P_1}{Q_1 + \cfrac{P_2}{\cfrac{y''}{y'''}}}$$

$$= \cdot \; \cdot \; \cdot \; \cdot \; \cdot$$

$$= Q_0 + \cfrac{P_1}{Q_1 + \cfrac{P_2}{Q_2 + \cfrac{P_3}{Q_3 + \cdots \cfrac{P_n}{Q_n + R_n}}}} ,$$

where $R_n = \dfrac{P_{n+1}}{\dfrac{y^{(n+1)}}{y^{(n+2)}}}$.

We will then have to consider the continued fraction

$$(5.6.4) \qquad \cfrac{1}{Q_0 + \cfrac{P_1}{Q_1 + \cfrac{P_2}{Q_2 + \cdots \cfrac{Q_n}{P_n + \cdots}}}} .$$

If (5.6.4) is a terminating continued fraction, it will, of course, be equal to y'/y. On the other hand, the question of convergence must be considered if the fraction is non-terminating. (See Ince, <u>Ordinary Differential Equations</u>, page 179). Ince gives the following example of a differential equation with a terminating continued fraction:

For

$$y = \frac{x}{m} y' + \frac{1}{m} y'' \qquad (m > 0),$$

we have

$$y' = \frac{x}{m-1} y'' + \frac{1}{m-1} y'''$$

$$y'' = \frac{x}{m-2} y''' + \frac{1}{m-2} y^{(iv)}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$y^{(n)} = \frac{x}{m-n} y^{(n+1)} + \frac{1}{m-n} y^{(n+2)}$$

where $n=1,2,\ldots,(m-1)$. When $n=m-1$, then

$$y^{(m-1)} = xy^{(m)} + y^{(m+1)} \; .$$

Differentiating, we obtain

$$y^{(m)} = xy^{(m+1)} + y^{(m)} + y^{(m+2)}$$

or

$$0 = xy^{(m+1)} + y^{(m+2)}.$$

It follows that

$$\frac{y'}{y} = \cfrac{m}{x+\cfrac{m-1}{x+\cfrac{m-2}{x+\cdots \cdot \frac{1}{x}}}} \; .$$

Since the fraction terminates, its value may be calculated by forming its successive approximants. We then have

$$\frac{y'}{y} = \frac{mx^{m-1}+(m-2)a_1x^{m-3}+(m-4)a_2x^{m-5}+\cdots}{x^m + a_1x^{m-2} + a_2x^{m-4}+\cdots} \quad ,$$

where

$$a_r = \frac{n!}{2^r r!(n-2r)!} \qquad r=1,2,3,\ldots \; .$$

Therefore

$$y = x^m + a_1 x^{m-2} + a_2 x^{m-4} + \dots + a_r$$

when r is even and

$$y = x^m + a_1 x^{m-2} + a_2 x^{m-4} + \dots + a_r x$$

when r is odd.

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ABSTRACT

After a short history and introduction, which includes a geometric representation of a continued fraction, the general theory is considered in the light of the recurrence formulas, product of linear fractional transformations, convergence definitions, and equivalence transformations.

Simple continued fractions, fractions of the form

$$b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cdots}} \quad ,$$

are the main topic of the second chapter. Here the reader is offered a complete coverage of both finite and infinite simple continued fractions. The basic theorems which pertain to fractions of the finite form are developed and then extended to cover infinite simple continued fractions. From these theorems it is concluded that every irrational number can be uniquely expressed as an infinite simple continued fraction; that the error involved when a simple continued

fraction is approximated by its qth approximate can be calculated by the formula

$$\left| x - \frac{A_q}{B_q} \right| < \frac{1}{B_q B_{q+1}} < \frac{1}{B_q^2}$$

where x is the value of the continued fraction and A_q and B_q are respectively the qth numerator and qth denominator; and that a periodic simple continued fraction is equal to a quadratic surd and conversely, the simple continued fraction which represents a quadratic surd is periodic.

Conditions for the convergence of continued fractions of the form

(A.)
$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots}}$$

and those of the form

(B.)
$$b_0 + \cfrac{a_1}{b_1 - \cfrac{a_2}{b_2 - \cdots}}$$

are established, while convergence theorems covering the continued fraction

(C.)
$$\cfrac{1}{a_1 + ib_1 + \cfrac{1}{a_2 + ib_2 + \cdots}}$$

are presented to the reader without proof. It is shown that (A.) will converge if the series

$$\sum_{n=2}^{\infty} b_{n-1} \frac{a_n}{b_n} \quad ,$$

diverges; (B.) will converge to a limit not greater than one if

$$b_n \geq a_n + 1$$

for all <u>n</u>; and (C.) will converge if the series

$$\sum_{n=0}^{\infty} |a_n + ib_n|$$

diverges. In line with the convergence theorems, conditions for the irrationality of the limits of (A.) and (B.) are also considered.

With the associating of the theorem of Vincent on the separation of roots and Lagrange's method of root approximation, the complete solution for real roots of an algebraic equation is accomplished by continued fractions. Vincent's theorem and the rather simple method employed enables one to determine the number of positive and negative roots of an equation and to locate these roots between two integers. This process calls for the use of finite simple continued fractions. Once a root has been located, it can be

approximated by Lagrange's method which involved infinite simple continued fractions.

Formulas for the solution of a quadratic equation in terms of the same periodic continued fraction are developed along with a theorem proved by Galois which also applies to the roots of a quadratic equation.

Analytic continued fractions, fractions whose elements are analytic functions, are discussed in the last chapter. A method is evolved which enables one to transform the infinite series

$$u_0 + u_1 + \cdots + u_n + \cdots$$

into an equivalent continued fraction of the form

$$u_0 + \cfrac{u_1}{1 - \cfrac{u_2}{u_1 + u_2 - \cfrac{u_1 u_3}{u_2 + u_3 - \cdots}}} \quad .$$

From this continued fraction, expressions for the power series are obtained. By transforming the hypergeometric series, $F(a,b,c;z)$, into the continued fraction of Gauss, we acquire expressions for $\log(1+z)$, $(1+z)$, $\arctan z$, e^z, etc., in terms of continued fractions.

The continued fraction of Gauss also leads us to a method of summing the divergent power series

$$1 - az + a(a+1)z^2 - a(a+1)(a+2)z^3 + \dots \quad .$$

This series is shown to be equivalent to the improper integral

$$\int_0^\infty e^{-z}\frac{dx}{1+zx} \quad .$$

A relation between continued fractions and orthogonal polynomials is established when it is shown that the denominators of the approximants of the continued fraction

$$\cfrac{1}{C_1x+D_1-\cfrac{E_2}{C_2x+D_2-\cfrac{E_3}{C_3x+D_3-\ \cdot_{\ \cdot_{\ \cdot}}}}}$$

are orthogonal polynomials. An example which affords a means of calculating the Legendre polynomials by continued fractions is considered.

Finally, a solution of the differential equation

$$y = Q_0 y' + P_1 y''$$

is accomplished by continued fractions and by way of example, a solution of the equation

$$y = \frac{x}{m} y' + \frac{1}{m} y'' \qquad (m > 0)$$

is obtained.